

GENERATING SETS OF TROPICAL HEMISPACES

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ABSTRACT. In this paper we consider tropical hemispaces, defined as tropically convex sets whose complements are also tropically convex, and tropical semispaces, defined as maximal tropically convex sets not containing a given point. We characterize tropical hemispaces by means of generating sets, composed of points and rays, what we call (P, R) -representations. With each hemisphere we associate a matrix with coefficients in the completed tropical semiring, satisfying an extended rank-one condition. Our proof techniques are based on homogenization (lifting a convex set to a cone), and the relation between tropical hemispaces and semispaces.

1. INTRODUCTION

Max-plus algebra refers to the the semiring $\mathbb{R}_{\max,+}$ which is composed of the set $\mathbb{R} \cup \{-\infty\}$ endowed with the operation $\alpha \oplus \beta := \max(\alpha, \beta)$ as addition and the usual real numbers addition as multiplication $\alpha \otimes \beta := \alpha + \beta$. Thus, the neutral elements for addition and multiplication are $-\infty$ and 0 respectively.

The max-plus semiring is algebraically isomorphic to the **max-times semiring** $\mathbb{R}_{\max,\times}$, which is composed of the set $\mathbb{R}_+ = [0, +\infty)$ endowed with the operation $\alpha \oplus \beta := \max(\alpha, \beta)$ as addition and the usual real numbers product as multiplication $\alpha \otimes \beta := \alpha\beta$. Consequently, in the max-times semiring, 0 is the neutral element for addition and 1 is the neutral element for multiplication.

In this paper, it is convenient to consider both realizations at the same time, under the common notation \mathbb{T} . In other words, the reader can assume one of the models from the very beginning. This will be called **tropical algebra**. We will use 0 to denote the neutral element for addition, 1 to denote the neutral element for multiplication, and \mathbb{T}_+ to denote the set of all invertible elements with respect to the multiplication, i.e., all the elements of \mathbb{T} different from 0.

The space \mathbb{T}^n of n -dimensional vectors $x = (x_1, \dots, x_n)$, endowed naturally with the component-wise addition (denoted also by \oplus) and $\lambda x := (\lambda \otimes x_1, \dots, \lambda \otimes x_n)$ as the multiplication of a vector by a scalar, is a semimodule over \mathbb{T} . The element $(0, 0, \dots, 0)$ in \mathbb{T}^n is also denoted by 0, and it is the identity for \oplus .

In **tropical convexity**, one first defines the tropical segment joining the points $x, y \in \mathbb{T}^n$ as the set

$$(1) \quad \{\alpha x \oplus \beta y \in \mathbb{T}^n \mid \alpha \oplus \beta = 1\},$$

and then calls a set $\mathcal{C} \subseteq \mathbb{T}^n$ tropically convex if it contains the tropical segment joining any two of its points (see Figure 1 below for an illustration of tropical segments in dimension 2).

The interest in tropical convexity (also known as max-plus convexity, \mathbb{B} -convexity) is due to its obvious similarity with the traditional convex geometry, inspired by the Litvinov-Maslov correspondence principle [13] and applications of tropical dequantization procedures in algebraic geometry, e.g., [11],[15]. On the other hand, tropical polytopes are cellular complexes made from usual convex polytopes of special kind according to the Develin-Sturmfels cellular decomposition [7] (also appearing in [3]). As a special case, the chain structure of tropical segments (1) was described in [7, 16, 19].

In this paper, we investigate the **tropical hemispaces**. This object comes from the abstract convexity [21, 22], where it is used in the Kakutani Theorem to separate two convex sets from each other. The proof of Kakutani Theorem makes use of Zorn's Lemma (relying on the Pasch axiom, which holds both in tropical and usual convexity). Briec, Horvath and Rubinov specialized this result to \mathbb{B} -convexity (tropical convexity, max-plus convexity) in [1, 3].

2010 *Mathematics Subject Classification.* 14T05, 52A01, secondary: 16Y60.

Key words and phrases. tropical convexity, \mathbb{B} -convexity, abstract convexity, max-plus algebra, hemisphere, semispace, tropical halfspace, rank-one matrix.

A different approach is to start from the separation of a point from a closed convex set, as investigated in many works (e.g., Zimmermann [23], Litvinov et al. [14], Cohen et al. [5, 6], Develin and Sturmfels [7], Briec et al. [3]). This Hahn-Banach type result can be extended to the separation of several convex sets by an application of non-linear Perron-Frobenius theory, as in Gaubert and Sergeev [10]. Here the separation does not rely on Zorn's Lemma and is more constructive (possibly leading to an algorithmic solution in the case of polytopes), but on the cost of losing the generality of separation of **any** two or several convex sets, as stated in the Kakutani Theorem and its extensions [3].

In the Hahn-Banach techniques, tropically convex sets are separated by means of closed halfspaces, defined as sets of $x \in \mathbb{T}^n$ satisfying an inequality of the form $\bigoplus_j \gamma_j x_j \oplus \alpha \leq \bigoplus_i \beta_i x_i \oplus \delta$. As shown by Joswig [12], closed halfspaces are unions of several closed sectors, which are convex in both tropical and ordinary sense.

Briec and Horvath [2] proved that the topological closure of any tropical hemisphere is a closed halfspace (appearing in the tropical Hahn-Banach separation, described by Joswig [12], discussed above, etc). Hence closed halfspaces, with respect to general hemispaces, are “almost everything”. However, the border between a tropical hemisphere and its complement has a generally unknown intricate pattern, with some pieces belonging to one hemisphere and the rest to the other. This pattern was not revealed by Briec and Horvath [2], who were more interested in Hahn-Banach type separation results. The present paper, though describing the thin structure of the border only indirectly, might help to elucidate its intricate pattern.

The paper is organized as follows. Section 2 is occupied with preliminaries on generators and recessive rays of tropically convex sets, suggesting the natural concept of (P, R) -representations, which can be seen as a relaxation of the traditional approach adopted by Gaubert and Katz [8, 9], who represented closed tropically convex sets by means of extreme points and rays. In Section 3 we study tropical semispaces: maximal tropically convex sets not containing a given point. These sets were described in detail by Nitica and Singer [16, 17, 18], who showed that they are precisely the complements of closed sectors (which constitute closed tropical halfspaces, as mentioned above). We give a simple proof of this result exploiting homogenization (lifting tropically convex sets to tropical cones) and the multiorder principle of tropical convexity formulated for tropical cones [20]. Hemispaces appear here as unions of (in general, infinitely many) complements of semispaces, i.e., closed sectors of [12].

Section 4 contains the **main results** on hemispaces. The purpose of Subsect. 4.1 is to reduce general hemispaces to conic hemispaces (i.e., hemispaces being tropical cones). This aim is finally achieved in Theorem 4.7. Meanwhile we draw all hemispaces on the plane, see Figures 2 and 3, and study the recessive rays of general hemispaces. In view of Theorem 4.7, in Subsect. 4.2 we study conic hemispaces only. There we introduce the “ α -matrix”, whose coefficients stem from the borderlines between hemispaces on two-dimensional coordinate planes. We formulate an extended rank-one condition on this matrix in Theorem 4.10 and immediately prove that it holds for any couple of hemispaces. The rest of Subsect. 4.2 is mainly devoted to a more detailed description of generating sets of conic hemispaces, which is needed to prove that the condition of Theorem 4.10 is also sufficient. In Subsect. 4.3, we obtain a number of corollaries of the previous results. First we verify that closed hemispaces are closed halfspaces, a result of Briec and Horvath [2], see Theorem 4.19 and Corollary 4.21. The (P, R) -representation of general hemispaces is formulated in Theorem 4.23, obtained as a combination of Theorems 4.7 and 4.10.

2. PRELIMINARIES

2.1. Tropically convex sets: generation and homogenization. We start describing some relations between tropically convex sets and tropical cones.

In what follows, for any $m, n \in \mathbb{Z}$ with $m \leq n$, we denote the set $\{m, m+1, \dots, n\}$ by $[m, n]$, or simply by $[n]$ when $m = 1$. Moreover, the multiplicative inverse of $\lambda \in \mathbb{T}_+$ will be denoted by λ^{-1} . For $x \in \mathbb{T}^n$ we define the **support** of x by

$$\text{supp}(x) := \{i \in [n] \mid x_i \neq \mathbb{0}\}.$$

The set of the vectors $\{e^i \mid i \in [n]\} \subseteq \mathbb{T}^n$ defined by

$$e_j^i = \begin{cases} \mathbb{1} & \text{if } i = j \\ \mathbb{0} & \text{if } i \neq j \end{cases}$$

form the **standard basis** in \mathbb{T}^n . We will refer to these vectors as the **unit vectors**.

Definition 2.1. A set $\mathcal{V} \subseteq \mathbb{T}^n$ is called a **tropical cone** if it is closed under (tropical) addition and multiplication by scalars.

For $P, R \subseteq \mathbb{T}^n$, we define

$$\text{conv}(P) := \left\{ \bigoplus_{y \in P} \lambda_y y \mid \lambda_y \in \mathbb{T} \text{ for } y \in P \text{ and } \bigoplus_{y \in P} \lambda_y = \mathbb{1} \right\}$$

and

$$\text{span}(R) := \left\{ \bigoplus_{y \in R} \lambda_y y \mid \lambda_y \in \mathbb{T} \text{ for } y \in R \right\},$$

where in both cases only a finite number of the scalars λ_y is not equal to $\mathbb{0}$.

Definition 2.2. Let $P, R \subseteq \mathbb{T}^n$. We say that a tropically convex set $\mathcal{C} \subseteq \mathbb{T}^n$ is **generated** by the pair (P, R) if

$$(2) \quad \mathcal{C} = \text{conv}(P) \oplus \text{span}(R).$$

For each tropically convex set $\mathcal{C} \subseteq \mathbb{T}^n$ at least one representation of the form (2) exists: just take $P = \mathcal{C}$ and $R = \emptyset$. A canonical representation of the form (2) can be written for closed tropically convex sets, by the tropical analogue of Minkowski theorem, due to Gaubert and Katz [8, 9].

Definition 2.3. For $C \subseteq \mathbb{T}^n$, the set

$$(3) \quad V_C = \{(\lambda, \lambda x_1, \dots, \lambda x_n) \mid (x_1, \dots, x_n) \in C, \lambda \in \mathbb{T}\} \subset \mathbb{T}^{n+1}$$

is called the **homogenization** of C .

Remark 2.4. If $\mathcal{C} \subseteq \mathbb{T}^n$ is a tropically convex set, then its homogenization $V_C \subseteq \mathbb{T}^{n+1}$ is a tropical cone. The coordinates in the homogenization are denoted by (x_0, x_1, \dots, x_n) .

Reversing the homogenization means taking a section of a tropical cone by a coordinate plane. Below we take only sections by $x_0 = \alpha$ (mostly with $\alpha = \mathbb{1}$), and not by $x_i = \alpha$ with $i \in [n]$. Hence we do not indicate the coordinate in the notation.

Definition 2.5. For $V \subseteq \mathbb{T}^{n+1}$ and $\alpha \in \mathbb{T}$, the set

$$(4) \quad C_V^\alpha = \{x \in \mathbb{T}^n \mid (\alpha, x) \in V\}$$

is called a **coordinate section** of V .

Proposition 2.6. If $\mathcal{V} \subseteq \mathbb{T}^{n+1}$ is expressed as $\text{span}(X)$ and its coordinate section $C_V^\mathbb{1}$ is nonempty, then $C_V^\mathbb{1} = \text{conv}(P_X) \oplus \text{span}(R_X)$ where

$$(5) \quad P_X := \{y \in \mathbb{T}^n \mid \exists \lambda \neq \mathbb{0} \text{ such that } (\lambda, \lambda y) \in X\} \text{ and } R_X := \{z \in \mathbb{T}^n \mid (\mathbb{0}, z) \in X\}.$$

Proof. If $x \in C_V^\mathbb{1}$ then

$$(6) \quad (\mathbb{1}, x) = \bigoplus_{(\mu_y, \mu_y y) \in X} \lambda_y (\mathbb{1}, y) \oplus \bigoplus_{(\mathbb{0}, z) \in X} \lambda_z (\mathbb{0}, z)$$

for some $\lambda_y, \lambda_z \in \mathbb{T}$ and $\mu_y \in \mathbb{T}$, with $\mu_y \neq \mathbb{0}$ and a finite number of λ_y, λ_z not equal to $\mathbb{0}$. As $\bigoplus \lambda_y = \mathbb{1}$, it follows that $x \in \text{conv}(P_X) \oplus \text{span}(R_X)$. Conversely, if $x \in \text{conv}(P_X) \oplus \text{span}(R_X)$ then $(\mathbb{1}, x)$ can be represented as in (6), hence $(\mathbb{1}, x) \in \mathcal{V}$ and $x \in C_V^\mathbb{1}$. \square

2.2. Recessive rays. We will use the following notions of recessive rays of a tropically convex set:

Definition 2.7. Let $\mathcal{C} \subseteq \mathbb{T}^n$ be a tropically convex set.

(i) Given $x \in \mathcal{C}$, the set of **recessive rays at x** , or **locally recessive rays at x** , is defined as

$$\text{rec}_x \mathcal{C} := \{z \in \mathbb{T}^n \mid x \oplus \lambda z \in \mathcal{C} \text{ for all } \lambda \in \mathbb{T}\}.$$

(ii) The set of **globally recessive rays** of \mathcal{C} , denoted by $\text{rec} \mathcal{C}$, consists of the rays that are recessive at each point of \mathcal{C} .

(iii) Given $i \in [n]$, the set of **i -recessive rays** of \mathcal{C} is defined as

$$\text{rec}_i \mathcal{C} := \{z \in \mathbb{T}^n \mid i \in \text{supp}(z) \text{ and there exists } \alpha \neq 0 \text{ such that } \alpha e^i \in \mathcal{C} \text{ and } z \in \text{rec}_{\alpha e^i} \mathcal{C}\}.$$

Note that if $\mathcal{C} = \text{conv}(P) \oplus \text{span}(R)$ as in (2), then $R \subseteq \text{rec} \mathcal{C}$.

The following lemma gives a general relation between recessive rays at different points of a tropically convex set \mathcal{C} .

Lemma 2.8. Let $\mathcal{C} \subseteq \mathbb{T}^n$ be a tropically convex set and $x, y \in \mathcal{C}$.

(i) If $\text{supp}(x) \subseteq \text{supp}(y)$ and $z \in \text{rec}_x \mathcal{C}$, then $z \in \text{rec}_y \mathcal{C}$. In particular, $\text{rec}_0 \mathcal{C} \subseteq \text{rec} \mathcal{C}$.

(ii) If $\text{supp}(y) \subseteq \text{supp}(z)$ and $z \in \text{rec}_y \mathcal{C}$, then $\lambda z \in \mathcal{C}$ for all large enough λ , and $z \in \text{rec} \mathcal{C}$.

Proof. (i) We need to prove that $y \oplus \lambda z \in \mathcal{C}$ for all $\lambda \in \mathbb{T}$. Since $\text{supp}(x) \subseteq \text{supp}(y)$ we have $\beta x \leq y$ for small enough β which we assume to satisfy $0 < \beta \leq 1$. Then, given $\lambda \in \mathbb{T}$, we conclude that

$$y \oplus \beta(x \oplus \beta^{-1} \lambda z) = y \oplus \lambda z \in \mathcal{C}$$

because $z \in \text{rec}_x \mathcal{C}$, and so the left hand side is a convex combination of two points in \mathcal{C} .

(ii) We have both $y \oplus \lambda z \in \mathcal{C}$ for all $\lambda \in \mathbb{T}$, since $z \in \text{rec}_y \mathcal{C}$, and $y \oplus \lambda z = \lambda z$ for all large enough λ , since $\text{supp}(y) \subseteq \text{supp}(z)$. Hence, $\lambda z \in \mathcal{C}$ for all large enough λ . In particular, given any $\beta \in \mathbb{T}$, there exists $\lambda > \beta$ such that $\lambda z \in \mathcal{C}$. Then, for any $x \in \mathcal{C}$, we have $x \oplus \beta z = x \oplus \beta \lambda^{-1} \lambda z \in \mathcal{C}$ because $\beta \lambda^{-1} \leq 1$. Thus, we conclude that $z \in \text{rec} \mathcal{C}$. \square

Corollary 2.9. If $\mathcal{C} \subseteq \mathbb{T}^n$ is a tropically convex set, then $\text{rec}_i \mathcal{C} \subseteq \text{rec} \mathcal{C}$ for all $i \in [n]$.

Proof. Apply Lemma 2.8 part (ii) with $y = \alpha e^i$, for some $\alpha \neq 0$ and $i \in \text{supp}(z)$ such that $z \in \text{rec}_{\alpha e^i} \mathcal{C}$. \square

Observe that $z \in \mathbb{T}^n$ can be globally recessive in \mathcal{C} but not i -recessive for any $i \in [n]$: in this case, \mathcal{C} does not contain any point of the form αe^i for $\alpha \neq 0$ and $i \in \text{supp}(z)$.

To introduce a topology we need to specialize \mathbb{T} to one of the models. Namely, if $\mathbb{T} = \mathbb{R}_{\max, \times}$ then we use the topology induced in \mathbb{R}_+^n by the usual Euclidean topology in the real space. If $\mathbb{T} = \mathbb{R}_{\max, +}$, then our topology is induced by the metric $d_\infty(x, y) = \max_{i \in [n]} |e^{x_i} - e^{y_i}|$.

For closed tropically convex sets, every locally recessive ray is globally recessive.

Proposition 2.10 (Gaubert and Katz [9]). *If a tropically convex set \mathcal{C} is closed, then $\text{rec}_x \mathcal{C} \subseteq \text{rec} \mathcal{C}$ for all $x \in \mathcal{C}$.*

Theorem 2.11. Let $\{\mathcal{C}_\ell\}$ be a family of tropically convex sets in \mathbb{T}^n generated by the pairs (P_ℓ, R_ℓ) :

$$\mathcal{C}_\ell = \text{conv}(P_\ell) \oplus \text{span}(R_\ell),$$

and let $\mathcal{C} := \text{conv}(\cup_\ell \mathcal{C}_\ell)$. Then,

$$(7) \quad \mathcal{C} = \text{conv}(\cup_\ell P_\ell) \oplus \text{span}(\cup_\ell R_\ell)$$

if any of the following conditions hold:

- (i) $R_\ell \subseteq \text{rec} \mathcal{C}$ for all ℓ ;
- (ii) \mathcal{C} is closed;
- (iii) For any $z \in R_\ell$ there exists $y \in \text{conv}(P_\ell)$ such that $\text{supp}(y) \subseteq \text{supp}(z)$;
- (iv) All points in P_ℓ have the same support for all ℓ .

Proof. It can be shown that in general we have

$$\mathcal{C} \subseteq \text{conv}(\cup_{\ell} P_{\ell}) \oplus \text{span}(\cup_{\ell} R_{\ell}).$$

(i) In this case $\mathcal{C} \oplus \text{span}(\cup_{\ell} R_{\ell}) \subseteq \mathcal{C}$, hence we also have the opposite inclusion

$$\text{conv}(\cup_{\ell} P_{\ell}) \oplus \text{span}(\cup_{\ell} R_{\ell}) \subseteq \mathcal{C}.$$

Let us now prove that $R_{\ell} \subseteq \text{rec } \mathcal{C}$ holds for cases (ii)-(iv).

(ii) Each $z \in R_{\ell}$ is recessive at all $y \in P_{\ell}$, hence by Proposition 2.10 it is globally recessive.

(iii) In this case any $z \in R_{\ell}$ satisfies the condition of Lemma 2.8 part (ii) for some point $y \in P_{\ell}$, hence it belongs to $\text{rec } \mathcal{C}$.

(iv) In this case points in P_{ℓ} have the minimal support in \mathcal{C} , and as the rays of R_{ℓ} are recessive at the points of P_{ℓ} , by Lemma 2.8 part (i) they are recessive at all points of \mathcal{C} . \square

3. SEMISPACES

In this section we aim to give a simpler proof for the structure of tropical semispaces, originally described by Nitica and Singer [16, 17], and to introduce tropical hemispaces with some preliminary results on their relation with semispaces.

3.1. Conic semispaces and sectors. We begin by recalling the definition of conic hemispaces.

Definition 3.1. A pair of tropical cones $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ is called a **couple of conic hemispaces** if $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$ and $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathbb{T}^n$.

For any $y \in \mathbb{T}^n$ and $i \in \text{supp}(y)$, define the following sets:

$$(8) \quad \mathcal{W}_i(y) := \left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in \text{supp}(y)} x_j y_j^{-1} \leq x_i y_i^{-1}, \text{ and } x_j = 0 \text{ for all } j \notin \text{supp } y \right\},$$

which will be referred to as **conic sectors** of type i . Since the complement of $\mathcal{W}_i(y)$

$$(9) \quad \mathbb{C}\mathcal{W}_i(y) = \left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in \text{supp}(y)} x_j y_j^{-1} > x_i y_i^{-1}, \text{ or } x_j > 0 \text{ for some } j \notin \text{supp } y \right\},$$

it follows that $\mathcal{W}_i(y)$ and $\mathbb{C}\mathcal{W}_i(y) \cup \{0\}$ are both tropical cones, so they form a couple of conic hemispaces. Also note that $y \in \mathcal{W}_i(y)$ for all $i \in \text{supp}(y)$.

The following result appears in several places ([4, 7, 12, 9, 20]).

Theorem 3.2. Let $\mathcal{V} \subseteq \mathbb{T}^n$ be a tropical cone and take $y \neq 0$ in \mathbb{T}^n . Then $y \in \mathcal{V}$ if and only if the set $\mathcal{W}_i(y) \setminus \{0\}$ contains a point from \mathcal{V} for each $i \in \text{supp}(y)$.

Proof. Note first that if $y \in \mathcal{V}$, then $y \in \mathcal{W}_i(y)$ for all $i \in \text{supp}(y)$.

Assume now that $x^i \neq 0$ is in $\mathcal{W}_i(y) \cap \mathcal{V}$, for $i \in \text{supp}(y)$. Since $x^i \in \mathcal{W}_i(y)$ and $x^i \neq 0$, we have $x_i^i \neq 0$ and $y_i x_j^i \leq y_j x_i^i$ for all $j \in [n]$. Then, y can be written as a tropical linear combination of the x^i 's:

$$y = \bigoplus_{i \in \text{supp}(y)} \lambda_i x^i,$$

where $\lambda_i = y_i (x_i^i)^{-1}$, therefore $y \in \mathcal{V}$. \square

Restating Theorem 3.2 we get the following.

Theorem 3.3. Let $\mathcal{V} \subseteq \mathbb{T}^n$ be a tropical cone and take $y \neq 0$ in \mathbb{T}^n . Then $y \notin \mathcal{V}$ if and only if $\mathcal{V} \subseteq \mathbb{C}\mathcal{W}_i(y) \cup \{0\}$ for some $i \in \text{supp}(y)$.

We are also interested in the following object.

Definition 3.4. A tropical cone in \mathbb{T}^n is called a **conic semispace** at point $y \neq 0$ in \mathbb{T}^n if it is a maximal tropical cone not containing y .

Corollary 3.5. *There are exactly the cardinality of $\text{supp}(y)$ conic semispaces at the point $y \neq 0$ of \mathbb{T}^n . These are given by the tropical cones $\mathbb{C}\mathcal{W}_i(y) \cup \{0\}$ for $i \in \text{supp}(y)$.*

Proof. Suppose that \mathcal{V} is a conic semispace at y . Since it is a tropical cone not containing y , Theorem 3.3 implies that it is contained in $\mathbb{C}\mathcal{W}_i(y) \cup \{0\}$ for some $i \in \text{supp}(y)$. By maximality, it follows that it coincides with $\mathbb{C}\mathcal{W}_i(y) \cup \{0\}$. \square

This statement shows that Theorem 3.3 is an instance of a separation theorem in abstract convexity. In particular, we obtain the following result.

Corollary 3.6. *Each nontrivial tropical cone \mathcal{V} can be represented as the intersection of the conic semispaces $\mathbb{C}\mathcal{W}_i(y) \cup \{0\}$ containing it (where $y \notin \mathcal{V}$ and $i \in \text{supp}(y)$), and for each complement F of a tropical cone, $F \cup \{0\}$ can be represented as the union of the conic sectors $\mathcal{W}_i(y)$ contained in $F \cup \{0\}$ (where $y \in F$ and $i \in \text{supp}(y)$).*

Lemma 3.7. *Assume that $x, y \in \mathbb{T}^n$ satisfy $\text{supp}(x) \cap \text{supp}(y) \neq \emptyset$. Then, for any $i \in \text{supp}(x) \cap \text{supp}(y)$, the nonzero point z with coordinates*

$$(10) \quad z_j := \min \{x_i^{-1}x_j, y_i^{-1}y_j\}$$

belongs to both $\mathcal{W}_i(x)$ and $\mathcal{W}_i(y)$.

Proof. Note that $z_j = 0$ for $j \notin \text{supp}(x) \cap \text{supp}(y)$. Moreover, since $z_i = 1$, we have $z_j x_j^{-1} \leq x_i^{-1} \leq z_i x_i^{-1}$ for all $j \in [n]$. Then, we conclude that $z \in \mathcal{W}_i(x)$. Similarly, it can be shown that $z \in \mathcal{W}_i(y)$. \square

Corollary 3.6 and Lemma 3.7 imply the following (preliminary) result on conic hemispaces.

Theorem 3.8. *For any couple of conic hemispaces \mathcal{H}_1 and \mathcal{H}_2 there exist disjoint subsets $I, J \subseteq [n]$ and a set $Y \subseteq \mathbb{T}^n$ such that*

$$(11) \quad \begin{aligned} \mathcal{H}_1 &= \text{span}\{\cup \mathcal{W}_i(y) \mid \mathcal{W}_i(y) \subseteq \mathcal{H}_1, i \in I, y \in Y \cap \mathcal{H}_1\}, \\ \mathcal{H}_2 &= \text{span}\{\cup \mathcal{W}_j(y) \mid \mathcal{W}_j(y) \subseteq \mathcal{H}_2, j \in J, y \in Y \cap \mathcal{H}_2\}. \end{aligned}$$

Proof. As \mathcal{H}_1 and \mathcal{H}_2 are complements of tropical cones, Corollary 3.6 yields that they are unions of the conic sectors contained in them. Since they are cones, these unions coincide with their conic hulls (i.e., spans). By Lemma 3.7, the sectors in \mathcal{H}_1 and \mathcal{H}_2 should be of different type, otherwise \mathcal{H}_1 and \mathcal{H}_2 have a nontrivial common point. \square

3.2. General semispaces and sectors. We now turn to tropically convex sets using the homogenization technique. Below we will be interested in the following objects.

Definition 3.9. *A pair of tropically convex sets $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ is called a couple of hemispaces if $\mathcal{H}_1 \cap \mathcal{H}_2 = \emptyset$ and $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathbb{T}^n$.*

If \mathbb{T}^n is viewed as the coordinate section of \mathbb{T}^{n+1} by $x_0 = 1$, then the coordinate sections of $\mathcal{W}_i(1, y)$ and $\mathbb{C}\mathcal{W}_i(1, y)$ for $y \in \mathbb{T}^n$ and $i \in \text{supp}(y) \cup \{0\} \subseteq [0, n]$ are

$$\begin{aligned} \mathcal{S}_0(y) &:= C_{\mathcal{W}_0(1, y)}^1 = \left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in \text{supp}(y)} x_j y_j^{-1} \leq 1 \text{ and } x_j = 0 \text{ for all } j \notin \text{supp}(y) \right\} \\ \mathcal{S}_i(y) &:= C_{\mathcal{W}_i(1, y)}^1 = \left\{ x \in \mathbb{T}^n \mid 1 \oplus \bigoplus_{j \in \text{supp}(y)} x_j y_j^{-1} \leq x_i y_i^{-1} \text{ and } x_j = 0 \text{ for all } j \notin \text{supp}(y) \right\}, \end{aligned}$$

and

$$\begin{aligned} \mathbb{C}\mathcal{S}_0(y) &= C_{\mathbb{C}\mathcal{W}_0(1, y)}^1 = \left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in \text{supp}(y)} x_j y_j^{-1} > 1 \text{ or } x_j > 0 \text{ for some } j \notin \text{supp}(y) \right\} \\ \mathbb{C}\mathcal{S}_i(y) &= C_{\mathbb{C}\mathcal{W}_i(1, y)}^1 = \left\{ x \in \mathbb{T}^n \mid 1 \oplus \bigoplus_{j \in \text{supp}(y)} x_j y_j^{-1} > x_i y_i^{-1} \text{ or } x_j > 0 \text{ for some } j \notin \text{supp}(y) \right\}. \end{aligned}$$

The sets $\mathcal{S}_i(y)$ will be called **sectors** of type i (see Figure 1 below for an illustration of sectors in dimension 2). Observe that both $\mathcal{S}_i(y)$ and $\mathbb{L}\mathcal{S}_i(y)$ are tropically convex sets and complements of each other, hence they are hemispaces.

Remark 3.10. We observe that the notation for sectors and semispaces is reversed as compared to the notation in Nitica and Singer [16, 17, 18].

Theorem 3.11. Let $y \in \mathbb{T}^n$ and let $\mathcal{C} \subseteq \mathbb{T}^n$ be tropically convex. Then $y \in \mathcal{C}$ if and only if $\mathcal{S}_i(y)$ contains a point in \mathcal{C} for $i = 0$ and for each $i \in \text{supp}(y)$.

Proof. Consider the homogenization $V_{\mathcal{C}}$ of \mathcal{C} , where each point $y \in \mathbb{T}^n$ is lifted to $(\mathbb{1}, y) \in \mathbb{T}^{n+1}$. In particular, each point $z \in V_{\mathcal{C}} \setminus \{0\}$ has $z_0 \neq 0$.

If the condition of the theorem is satisfied for some $y \in \mathbb{T}^n$, i.e. if for $i = 0$ and each $i \in \text{supp}(y)$ there exist $x^i \in \mathcal{S}_i(y) \cap \mathcal{C}$, then the condition of Theorem 3.2 is satisfied for $(\mathbb{1}, y)$ with $(\mathbb{1}, x^i) \in \mathcal{W}_i(y) \cap V_{\mathcal{C}}$. It follows that $(\mathbb{1}, y) \in V_{\mathcal{C}}$ and $y \in \mathcal{C}$.

Conversely, $y \in \mathcal{C}$ implies $z := (\mathbb{1}, y) \in V_{\mathcal{C}}$, hence we can find $z^i \in (\mathcal{W}_i(z) \setminus \{0\}) \cap V_{\mathcal{C}}$ for all $i \in \text{supp}(y)$ and $i = 0$. Since $z_0^i \neq 0$ and the conic sectors $\mathcal{W}_i(z)$ are tropical cones, we can assume that $z_0^i = \mathbb{1}$ so that $z^i = (\mathbb{1}, x^i)$, where $x^i \in \mathcal{S}_i(y) \cap \mathcal{C}$. It follows that each $\mathcal{S}_i(y) \cap \mathcal{C}$ contains a point. \square

Theorem 3.11 is equivalent to the following.

Theorem 3.12. Let $y \in \mathbb{T}^n$, and let $\mathcal{C} \subseteq \mathbb{T}^n$ be a tropically convex set. Then $y \notin \mathcal{C}$ if and only if $\mathcal{C} \subseteq \mathbb{L}\mathcal{S}_i(y)$ for $i = 0$ or some $i \in \text{supp}(y)$.

Definition 3.13. A tropically convex set in \mathbb{T}^n is called a **semispace** at point $y \in \mathbb{T}^n$ if it is a maximal tropically convex set not containing y .

Corollary 3.14. There are exactly the cardinality of $\text{supp}(y)$ plus one semispaces at the point $y \in \mathbb{T}^n$. These are given by the tropically convex sets $\mathbb{L}\mathcal{S}_i(y)$ for $i = 0$ and $i \in \text{supp}(y)$.

Proof. By analogy with Corollary 3.5. \square

We also have the following version of Corollary 3.6.

Corollary 3.15. Each tropically convex set \mathcal{C} can be represented as the intersection of the semispaces $\mathbb{L}\mathcal{S}_i(y)$ containing it (where $y \notin \mathcal{C}$ and $i = 0$ or $i \in \text{supp}(y)$), and each complement F of a tropically convex set can be represented as the union of the sectors $\mathcal{S}_i(y)$ contained in F (where $y \in F$ and $i = 0$ or $i \in \text{supp}(y)$).

Lemma 3.16. For any two points $x, y \in \mathbb{T}^n$ and $i = 0$ or $i \in \text{supp}(x) \cap \text{supp}(y)$ the intersection $\mathcal{S}_i(x) \cap \mathcal{S}_i(y)$ is non-empty.

Proof. Consider the points $(\mathbb{1}, x)$ and $(\mathbb{1}, y)$ and observe that for $\mathcal{V} := \mathcal{W}_i(\mathbb{1}, x) \cap \mathcal{W}_i(\mathbb{1}, y)$, the section $C_{\mathcal{V}}^{\mathbb{1}}$ is precisely $\mathcal{S}_i(x) \cap \mathcal{S}_i(y)$. For $i = 0$ or $i \in \text{supp}(x) \cap \text{supp}(y)$ a nontrivial point $z \in \mathcal{V}$ can be constructed using (10). Note that $z_0 = \min\{x_i^{-1}, y_i^{-1}\} \neq 0$ and hence this point can be translated to a point in $C_{\mathcal{V}}^{\mathbb{1}}$. \square

Corollary 3.15 and Lemma 3.16 imply the following (preliminary) result on general hemispaces.

Theorem 3.17. For any couple of hemispaces \mathcal{H}_1 and \mathcal{H}_2 there exist disjoint subsets $I, J \subseteq [0, n]$ and $Y \subseteq \mathbb{T}^n$ such that

$$(12) \quad \begin{aligned} \mathcal{H}_1 &= \text{conv}(\{\cup \mathcal{S}_i(y) \mid \mathcal{S}_i(y) \subseteq \mathcal{H}_1, i \in I, y \in Y \cap \mathcal{H}_1\}), \\ \mathcal{H}_2 &= \text{conv}(\{\cup \mathcal{S}_j(y) \mid \mathcal{S}_j(y) \subseteq \mathcal{H}_2, j \in J, y \in Y \cap \mathcal{H}_2\}). \end{aligned}$$

Proof. By analogy with Theorem 3.8, using Corollary 3.15 and Lemma 3.16 instead of their conic versions. \square

4. HEMISPACES

4.1. Generating sets and homogenization. Theorem 3.17 is suggesting to unite the generating sets of all the sectors $\mathcal{S}_i(y)$ contained in a hemispace \mathcal{H} . However, this can be legitimately done only in the situations described by Theorem 2.11.

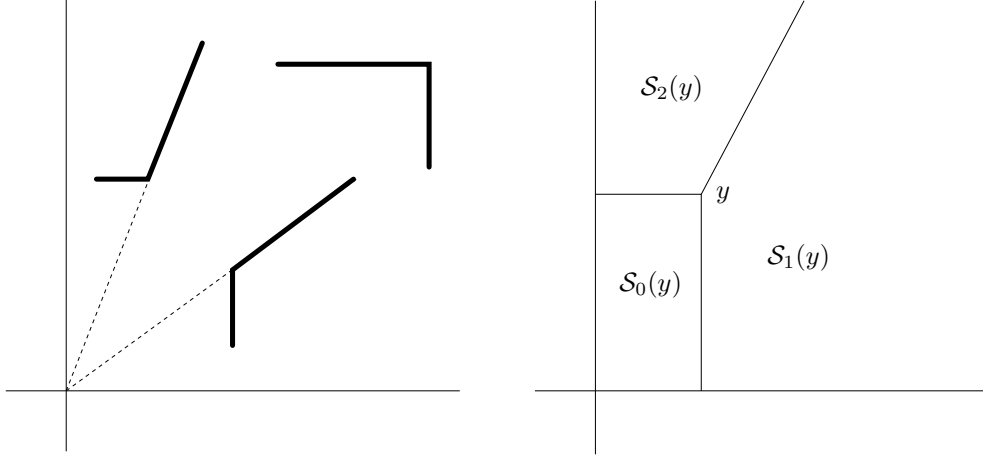


FIGURE 1. Tropical (max-times) segments (on the left) and sectors based at a point y with full support (on the right) in dimension 2.

Lemma 4.1. *The sectors $\mathcal{S}_0(y)$, $\mathcal{S}_i(y)$, and $\mathcal{W}_i(y)$ for $i \in \text{supp}(y)$ can be represented as*

$$\begin{aligned}
 \mathcal{S}_0(y) &= \text{conv}(\{0, y_j e^j \mid j \in \text{supp}(y)\}), \\
 \mathcal{S}_i(y) &= \{y_i e^i\} \oplus \text{span}(\{e^i \oplus y_j y_i^{-1} e^j \mid j \in \text{supp}(y)\}), \\
 \mathcal{W}_i(y) &= \text{span}(\{e^i \oplus y_j y_i^{-1} e^j \mid j \in \text{supp}(y)\}).
 \end{aligned}
 \tag{13}$$

Proof. Observe that $\mathcal{W}_i(y)$ consists of the vectors $x \in \mathbb{T}^n$ satisfying $y_i y_j^{-1} x_j \leq x_i$ for all $j \in \text{supp}(y)$ and $x_j = 0$ otherwise. Thus, if $x \in \mathcal{W}_i(y)$, we have $x = \bigoplus_{j \in \text{supp}(y)} y_i y_j^{-1} x_j (e^i \oplus y_j y_i^{-1} e^j)$ since $i \in \text{supp}(y)$. It follows that $\mathcal{W}_i(y) \subseteq \text{span}(\{e^i \oplus y_j y_i^{-1} e^j \mid j \in \text{supp}(y)\})$. Moreover, since the vector $e^i \oplus y_j y_i^{-1} e^j$ belongs to $\mathcal{W}_i(y)$ for any $j \in \text{supp}(y)$, we conclude that $\text{span}(\{e^i \oplus y_j y_i^{-1} e^j \mid j \in \text{supp}(y)\}) \subseteq \mathcal{W}_i(y)$.

Note that the (P, R) -representations for $\mathcal{S}_0(y)$ and $\mathcal{S}_i(y)$ can be obtained as those of sections of $\mathcal{W}_i(y)$ by coordinate planes, i.e., by means of Proposition 2.6. \square

By Theorem 2.11 part (iv), to obtain convex hulls of sectors of the same type, based at the points y of some set $Y \subseteq \mathbb{T}^n$, for $i = 0$ or $i \in [n]$, we can gather the generators and recessive rays of (13) as follows:

$$\begin{aligned}
 \text{conv}(\cup_{y \in Y} \mathcal{S}_0(y)) &= \text{conv}(\cup_{y \in Y, j \in [n]} \{0, y_j e^j\}), \\
 \text{conv}(\cup_{y \in Y} \mathcal{S}_i(y)) &= \text{conv}(\cup_{y \in Y} \{y_i e^i\}) \oplus \text{span}(\cup_{y \in Y, j \in [n]} \{e^i \oplus y_j y_i^{-1} e^j\}), \\
 \text{conv}(\cup_{y \in Y} \mathcal{W}_i(y)) &= \text{span}(\cup_{y \in Y, j \in [n]} \{e^i \oplus y_j y_i^{-1} e^j\}).
 \end{aligned}
 \tag{14}$$

These representations already yield all tropical hemispaces on the plane. Indeed, in the case $n = 2$, applying Theorems 3.8 and 3.17 we see that at least one of the sets I or J consists of only one index, and we get a (P, R) -representation of the corresponding hemispace exactly as in (14). By careful inspection of all possible cases we obtain the sets shown on the diagrams of Figures 2 and 3. Using the form of typical tropical segments on the plane, shown on the left-hand side of Figure 1, it is easy to check graphically that all these sets and their complements are indeed tropically convex sets (and hence, indeed, hemispaces). All figures are done in the max-times semiring $\mathbb{R}_{\max, \times}$.

Theorem 4.2. *For any hemispace $\mathcal{H} \subset \mathbb{T}^n$, a (P, R) -representation can be obtained by uniting the (P, R) -representations (13) of all $\mathcal{S}_i(y) \subseteq \mathcal{H}$, or $\mathcal{W}_i(y) \subseteq \mathcal{H}$ when \mathcal{H} is conic. In the resulting (P, R) -representation, P consists of multiples of unit vectors and possibly 0 (containing only 0 if \mathcal{H} is conic), while R consists of unit vectors and combinations of two unit vectors.*

Proof. By Theorem 3.17 any hemispace \mathcal{H} can be represented as the tropical convex hull of all the sectors $\mathcal{S}_i(y)$ contained in \mathcal{H} . By Lemma 4.1, the sectors $\mathcal{S}_i(y)$ have (P, R) -representations with $P = \{y_i e^i\}$ and R consisting of e^i and combinations of two unit vectors. Observe that these (P, R) -representations satisfy the

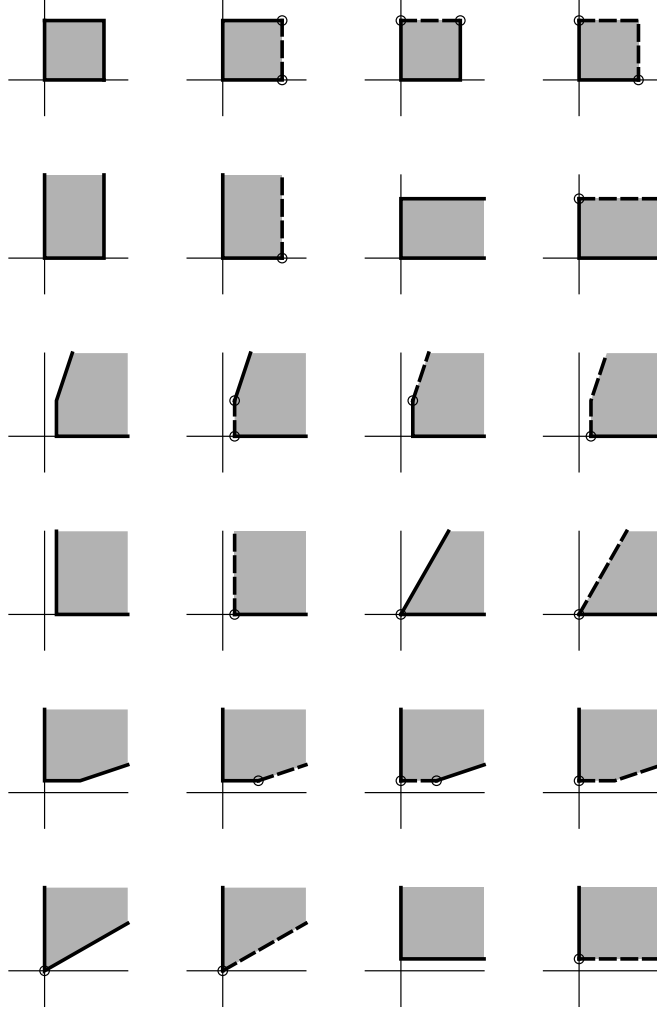


FIGURE 2. The tropical hemispaces in dimension 2 which can be obtained as unions of sectors of the same type based at points with full support.

condition of Theorem 2.11 part (iii), hence the claim. (Note that in the case of conic hemispaces gathering recessive rays of $\mathcal{W}_i(y)$ is straightforward.) \square

Remark 4.3. A set $X \subseteq \mathbb{T}^n$ is called *projectively closed* if the set $\{(\max_i x_i)^{-1}x \mid x \in X\}$ is closed. If a (P, R) -representation of a closed hemisphere is constructed as in Theorem 4.2, it may not have R projectively closed. For example, consider the hemisphere $\{(x_1, x_2) \in \mathbb{T}^2 \mid x_2 \geq \lambda\}$, where $\lambda \in \mathbb{T}_+$. Then, we have $P = \{\beta e^2 \mid \beta \geq \lambda\}$ and $R = \{e^2 \oplus \alpha e^1 \mid \alpha \in \mathbb{T}\}$.

The following three statements about recessive rays of hemispaces will not be useful in the sequel and can be skipped at a first reading (in this case, proceed with Theorem 4.7). Namely, we will investigate how recessive rays are distributed among hemispaces. Observe that if $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ is a couple of hemispaces, then \mathcal{H}_1 and \mathcal{H}_2 can have recessive rays in common. For example, if $\mathcal{H}_1 = \{(x_1, x_2) \in \mathbb{T}^2 \mid x_2 \geq \lambda\}$ and $\mathcal{H}_2 = \{(x_1, x_2) \in \mathbb{T}^2 \mid x_2 < \lambda\}$ for some $\lambda \in \mathbb{T}_+$, then e^1 is recessive in both \mathcal{H}_1 and \mathcal{H}_2 .

Proposition 4.4. Let $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ be a couple of hemispaces with $\mathbb{0} \in \mathcal{H}_1$. Each vector $z \in \mathbb{T}^n$ is either globally recessive in \mathcal{H}_1 , or i -recessive (hence also globally recessive) in \mathcal{H}_2 for some $i \in \text{supp}(z)$.

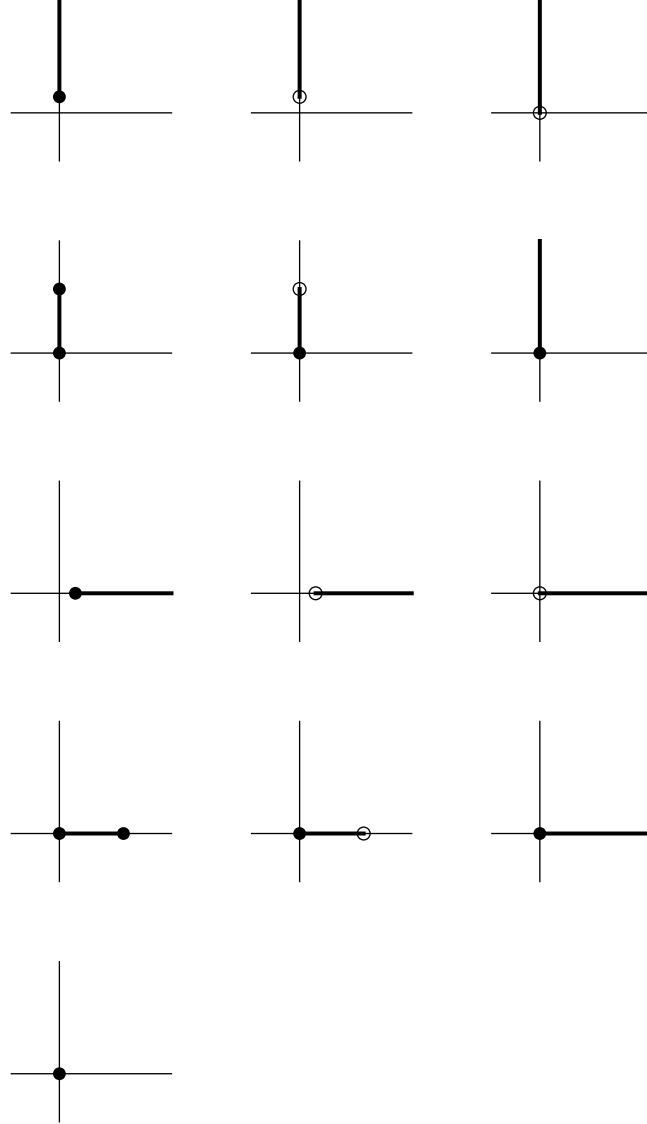


FIGURE 3. The tropical hemispaces in dimension 2 which can be obtained as unions of sectors of the same type based at points with non-full support.

Proof. Considering a line $\{\lambda z \mid \lambda \in \mathbb{T}\}$, we obtain that either the whole line belongs to \mathcal{H}_1 in which case $z \in \text{rec}_0 \mathcal{H}_1 = \text{rec } \mathcal{H}_1$, or there exists β such that $\{\lambda z \mid \lambda \geq \beta\} \subseteq \mathcal{H}_2$, in which case z is recessive in \mathcal{H}_2 at $\beta z \in \mathcal{H}_2$. For some $i \in \text{supp}(z)$ there should be $\alpha e^i \in \mathcal{H}_2$, otherwise all vectors with the support of z belong to \mathcal{H}_1 . Then, for $\lambda \geq \beta$ we have $\alpha e^i \oplus \lambda z \in \mathcal{H}_2$ (as a tropical convex combination of αe^i and λy), and for $\lambda < \beta$ we obtain

$$\alpha e^i \oplus \lambda z = \alpha e^i \oplus \lambda \beta^{-1} \beta z \in \mathcal{H}_2,$$

as a tropical convex combination of αe^i and βz .

We have shown that if z is not globally recessive in \mathcal{H}_1 , then it is i -recessive in \mathcal{H}_2 for some $i \in \text{supp}(z)$. On the other hand, if it is i -recessive in \mathcal{H}_2 then $\alpha e^i \oplus \lambda z = \lambda z \in \mathcal{H}_2$ for large enough λ , which means that $z \notin \text{rec } \mathcal{H}_1$ because $0 \in \mathcal{H}_1$. \square

The next results show that if \mathcal{H}_1 and \mathcal{H}_2 have a common recessive ray z , then each e^i for $i \in \text{supp}(z)$ is recessive in \mathcal{H}_1 and \mathcal{H}_2 .

Proposition 4.5. *Let z be a recessive ray of a hemispace \mathcal{H} and suppose that \mathcal{H} does not contain multiples of e^j for any $j \in \text{supp}(z)$. Then each e^j for $j \in \text{supp}(z)$ is a recessive ray of \mathcal{H} .*

Proof. Take $y \in \mathcal{H}$, then $\text{supp}(y) \not\subseteq \text{supp}(z)$, for otherwise \mathcal{H} would contain a multiple of e^j for some $j \in \text{supp}(z)$. Denote by M the set of indices i in $\text{supp}(y)$ such that \mathcal{H} contains a multiple of e^i . As $y \in \mathcal{H}$, the set M is non-empty, and by the above, $M \subseteq \text{supp}(y) \setminus \text{supp}(z)$. Denote by N the complement of M in $\text{supp}(y) \cup \text{supp}(z) = \text{supp}(y \oplus z)$.

Since \mathcal{H} does not contain multiples of e^j for any $j \in N$, there is a (P, R) -representation of \mathcal{H} as in Theorem 4.2, where e^j for $j \in N$ are not in P . Hence for any λ ,

$$(15) \quad y \oplus \lambda z = \bigoplus_{i \in M} \alpha_i(\lambda) e^i \oplus \bigoplus_{i \in M, j \in N} \beta_{ij}(\lambda) (e^i \oplus \gamma_{ij}(\lambda) e^j),$$

where $e^i \oplus \gamma_{ij}(\lambda) e^j \in R$ for all $i \in M, j \in N$ and all λ . Also $\beta_{ij}(\lambda) \leq y_i$ for all $i \in M$, hence $\gamma_{ij}(\lambda) \geq \lambda z_j y_i^{-1}$ for all $j \in \text{supp}(z)$. Then, for any $j \in \text{supp}(z)$, the point

$$y \oplus y_i (e^i \oplus \gamma_{ij}(\lambda) e^j) = y \oplus y_i \gamma_{ij}(\lambda) e^j,$$

where $i \in M$, belongs to \mathcal{H} and $y_i \gamma_{ij}(\lambda)$ can be made arbitrarily large. Hence, $e^j \in \text{rec}_y \mathcal{H}$. As we took an arbitrary $y \in \mathcal{H}$, the result follows. \square

Corollary 4.6. *Let $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ be a couple of hemispaces. If $z \in \text{rec } \mathcal{H}_1 \cap \text{rec } \mathcal{H}_2$, then also any e^j , for $j \in \text{supp}(z)$, belongs to $\text{rec } \mathcal{H}_1 \cap \text{rec } \mathcal{H}_2$.*

Proof. If $0 \in \mathcal{H}_1$, then Proposition 4.4 implies that z cannot be j -recessive in \mathcal{H}_2 for any $j \in \text{supp}(z)$. Therefore, \mathcal{H}_2 does not contain a multiple of e^j for any $j \in \text{supp}(z)$.

By Proposition 4.5 all e^j for $j \in \text{supp}(z)$ are recessive in \mathcal{H}_2 . However, they are not j -recessive in \mathcal{H}_2 , and hence by Proposition 4.4, they are also recessive in \mathcal{H}_1 . \square

We now show that a general hemispace can be obtained as a section of a conic hemispace.

Theorem 4.7. *For any couple of hemispaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ there exists a couple of conic hemispaces $\mathcal{V}_1, \mathcal{V}_2 \subseteq \mathbb{T}^{n+1}$ such that $\mathcal{H}_1 = C_{\mathcal{V}_1}^1$ and $\mathcal{H}_2 = C_{\mathcal{V}_2}^1$. More precisely, if (P_1, R_1) and (P_2, R_2) are respectively the representations of \mathcal{H}_1 and \mathcal{H}_2 given by Theorem 4.2, then the conic hemispaces \mathcal{V}_1 and \mathcal{V}_2 can be defined as:*

$$\mathcal{V}_1 := \text{span}(\{(\mathbb{1}, x) \mid x \in P_1\} \cup \{(\mathbb{0}, y) \mid y \in R_1\})$$

and

$$\mathcal{V}_2 := \text{span}(\{(\mathbb{1}, x) \mid x \in P_2\} \cup \{(\mathbb{0}, y) \mid y \in R_2\}).$$

Proof. In the first place, note that the tropical cones \mathcal{V}_1 and \mathcal{V}_2 are equivalently given by:

$$\mathcal{V}_1 = \{(\lambda, \lambda x) \mid x \in \mathcal{H}_1, \lambda \in \mathbb{T}\} \cup \text{span}(\{(\mathbb{0}, x) \mid x \in R_1\})$$

and

$$\mathcal{V}_2 = \{(\lambda, \lambda x) \mid x \in \mathcal{H}_2, \lambda \in \mathbb{T}\} \cup \text{span}(\{(\mathbb{0}, x) \mid x \in R_2\}).$$

Indeed, with both definitions \mathcal{V}_1 and \mathcal{V}_2 satisfy $\alpha \mathcal{H}_1 = C_{\mathcal{V}_1}^\alpha$ and $\alpha \mathcal{H}_2 = C_{\mathcal{V}_2}^\alpha$ for any $\alpha \neq 0$. For $\alpha = 0$, we have $C_{\mathcal{V}_1}^0 = \text{span}(R_1)$ and $C_{\mathcal{V}_2}^0 = \text{span}(R_2)$, with both definitions.

We next show that \mathcal{V}_1 and \mathcal{V}_2 are also a couple of conic hemispaces, i.e. $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$ and $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathbb{T}^{n+1}$.

With this aim, it is convenient to recall first that $\{e^i \oplus y_j y_i^{-1} e^j \mid j \in \text{supp}(y)\} \subseteq R_1$ (resp. $\{e^i \oplus y_j y_i^{-1} e^j \mid j \in \text{supp}(y)\} \subseteq R_2$) and $\{y_i e^i\} \subseteq P_1$ (resp. $\{y_i e^i\} \subseteq P_2$) if $\mathcal{S}_i(y) \subseteq \mathcal{H}_1$ (resp. $\mathcal{S}_i(y) \subseteq \mathcal{H}_2$) for some $y \in \mathbb{T}^n$ and $i \neq 0$.

Since \mathcal{H}_1 and \mathcal{H}_2 are a couple of hemispaces, it readily follows that

$$(16) \quad \{(\lambda, \lambda x) \mid x \in \mathcal{H}_1, \lambda \in \mathbb{T}\} \cup \{(\lambda, \lambda x) \mid x \in \mathcal{H}_2, \lambda \in \mathbb{T}\} = \{z \in \mathbb{T}^{n+1} \mid z_0 \neq 0\} \cup \{0\}$$

and

$$(17) \quad \{(\lambda, \lambda x) \mid x \in \mathcal{H}_1, \lambda \in \mathbb{T}\} \cap \{(\lambda, \lambda x) \mid x \in \mathcal{H}_2, \lambda \in \mathbb{T}\} = \{0\}.$$

Take now a vector $y \in \mathbb{T}^n$ and assume, without loss of generality, that $y \in \mathcal{H}_1$. By Theorem 3.12, and since \mathcal{H}_1 is a union of sectors by Theorem 3.17, it follows that $\mathcal{S}_i(y) \subset \mathcal{H}_1$ for $i = 0$ or for some $i \in \text{supp}(y)$.

If $\mathcal{S}_i(y) \subseteq \mathcal{H}_1$ for some $i \neq 0$, by (13) it follows that $(0, y) \in \text{span}(\{(0, x) \mid x \in R_1\})$. In the case when $\mathcal{S}_i(y) \not\subseteq \mathcal{H}_1$ for any $i \neq 0$, we have $\mathcal{S}_0(y) \subseteq \mathcal{H}_1$, and we look at αy for $\alpha \neq 0$.

If for some $\alpha \neq 0$ we have $\mathcal{S}_0(\alpha y) \not\subseteq \mathcal{H}_1$ and $\mathcal{S}_0(\alpha y) \not\subseteq \mathcal{H}_2$, then $\mathcal{S}_i(\alpha y) \subseteq \mathcal{H}_1$ or $\mathcal{S}_i(\alpha y) \subseteq \mathcal{H}_2$ for some $i \neq 0$, implying that $(0, y) \in \text{span}(\{(0, x) \mid x \in R_1\})$ or $(0, y) \in \text{span}(\{(0, x) \mid x \in R_2\})$.

We are left with the case when $\mathcal{S}_0(\alpha y) \subseteq \mathcal{H}_1$ or $\mathcal{S}_0(\alpha y) \subseteq \mathcal{H}_2$ for each α . Since the sets $\mathcal{S}_0(\alpha y)$ are increasing with α , it can be only that either $\mathcal{S}_0(\alpha y) \subseteq \mathcal{H}_1$ for all α , or $\mathcal{S}_0(\alpha y) \subseteq \mathcal{H}_2$ for all α . Assume the first case. Then, we obtain that all vectors x with $\text{supp}(x) \subseteq \text{supp}(y)$ are in \mathcal{H}_1 and hence $\mathcal{S}_i(y) \subseteq \mathcal{H}_1$ for all $i \in \text{supp}(y)$, implying that $(0, y) \in \text{span}(\{(0, x) \mid x \in R_1\})$. This shows that

$$\text{span}(\{(0, x) \mid x \in R_1\}) \cup \text{span}(\{(0, x) \mid x \in R_2\}) = \{z \in \mathbb{T}^{n+1} \mid z_0 = 0\},$$

and so $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathbb{T}^{n+1}$ by (16).

Finally, assume that $(0, y) \in \text{span}(\{(0, x) \mid x \in R_1\}) \cap \text{span}(\{(0, x) \mid x \in R_2\})$ and $y \neq 0$. Then, note that (13) implies that there exist $x^1 \in \mathcal{H}_1$ and $x^2 \in \mathcal{H}_2$ such that $\text{supp}(x^1) \subset \text{supp}(y)$ and $\text{supp}(x^2) \subset \text{supp}(y)$. Since $x^1 \oplus \lambda y = x^2 \oplus \lambda y = \lambda y$ for large enough λ and $y \in \text{span}(R_1) \cap \text{span}(R_2) \subseteq \text{rec } \mathcal{H}_1 \cap \text{rec } \mathcal{H}_2$, we have $\lambda y \in \mathcal{H}_1 \cap \mathcal{H}_2$ for large enough λ , a contradiction. Therefore,

$$\text{span}(\{(0, x) \mid x \in R_1\}) \cap \text{span}(\{(0, x) \mid x \in R_2\}) = \{0\}.$$

We conclude that $\mathcal{V}_1 \cap \mathcal{V}_2 = \{0\}$, since we also have $\{(\lambda, \lambda x) \mid x \in \mathcal{H}_2, \lambda \in \mathbb{T}\} \cap \text{span}(\{(0, x) \mid x \in R_1\}) = \{0\}$, $\{(\lambda, \lambda x) \mid x \in \mathcal{H}_1, \lambda \in \mathbb{T}\} \cap \text{span}(\{(0, x) \mid x \in R_2\}) = \{0\}$ and (17). \square

Remark 4.8. *The homogenization of a closed hemispace, constructed as in Theorem 4.7, may not be closed. Consider the example of Remark 4.3. In that case, \mathcal{V} is generated by $e^0 \oplus \lambda e^2$, e^2 and $e^2 \oplus \alpha e^1$ for $\alpha \in \mathbb{T}$. All multiples of e^1 are limiting points of \mathcal{V} , but they are not in \mathcal{V} . However, all limiting points x with nontrivial x_0 are in \mathcal{V} , and this is true also for general closed hemispaces.*

4.2. Generators of conic hemispaces. We know that a conic hemispace, as a conic hull of sectors, is generated by unit vectors and combinations of two unit vectors. Therefore, to describe a couple of hemispaces by their generating sets we need to understand how the combinations of two unit vectors are distributed among them. With this aim, we first associate with a couple of conic hemispaces $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ the index sets

$$(18) \quad I := \{i \in [n] \mid e^i \in \mathcal{H}_1\} \quad \text{and} \quad J := \{j \in [n] \mid e^j \in \mathcal{H}_2\}.$$

The following lemma is elementary and will rather serve to define below the coefficients α_{ij} . In what follows, for some purposes it will be convenient to assume that scalars can also take the value $+\infty$ (the structure which is obtained defining $\lambda \otimes (+\infty) := +\infty$ for $\lambda \in \mathbb{T}_+$ and $0 \otimes (+\infty) := 0$ is usually known as the completed semiring, see for instance [6]) and to adopt the convention $e^i \oplus \lambda e^j = e^j$ if $\lambda = +\infty$.

Lemma 4.9. *For any $i \in I$ and $j \in J$ we have*

$$\sup \{\alpha \in \mathbb{T} \cup \{+\infty\} \mid e^i \oplus \alpha e^j \in \mathcal{H}_1\} = \inf \{\beta \in \mathbb{T} \cup \{+\infty\} \mid e^i \oplus \beta e^j \in \mathcal{H}_2\}.$$

Proof. Observe that the lemma readily follows from the fact that if $e^i \oplus \beta e^j \in \mathcal{H}_2$ for some $\beta \neq +\infty$, then $e^i \oplus \gamma e^j \in \mathcal{H}_2$ for any $\gamma > \beta$ since $e^j \in \mathcal{H}_2$, and every combination of two unit vectors should belong either to \mathcal{H}_1 or to \mathcal{H}_2 . \square

Henceforth, the matrix whose entries are the coefficients

$$(19) \quad \alpha_{ij} := \sup \{\alpha \in \mathbb{T} \cup \{+\infty\} \mid e^i \oplus \alpha e^j \in \mathcal{H}_1\} = \inf \{\beta \in \mathbb{T} \cup \{+\infty\} \mid e^i \oplus \beta e^j \in \mathcal{H}_2\}$$

will be referred to as the α -**matrix** (associated with the couple of conic hemispaces $\mathcal{H}_1, \mathcal{H}_2$). Besides, with each coefficient α_{ij} we associate the pair of subsets of $\mathbb{T} \cup \{+\infty\}$ defined by

$$(20) \quad (\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)}) := \begin{cases} (\{\lambda \mid \lambda < \alpha_{ij}\}, \{\lambda \mid \lambda \geq \alpha_{ij}\}) & \text{if } \alpha_{ij} \in \mathbb{T}_+ \text{ and } e^i \oplus \alpha_{ij} e^j \in \mathcal{H}_2, \\ (\{\lambda \mid \lambda \leq \alpha_{ij}\}, \{\lambda \mid \lambda > \alpha_{ij}\}) & \text{if } \alpha_{ij} \in \mathbb{T}_+ \text{ and } e^i \oplus \alpha_{ij} e^j \in \mathcal{H}_1, \\ (\{\alpha_{ij}\}, \{\lambda \mid \lambda > \alpha_{ij}\}) & \text{if } \alpha_{ij} = 0, \\ (\{\lambda \mid \lambda < \alpha_{ij}\}, \{\alpha_{ij}\}) & \text{if } \alpha_{ij} = +\infty. \end{cases}$$

Thus, by Lemma 4.9 it follows that

$$(21) \quad \{e^i \oplus \lambda e^j \mid \lambda \in \alpha_{ij}^{(-)}\} \subset \mathcal{H}_1 \quad \text{and} \quad \{e^i \oplus \lambda e^j \mid \lambda \in \alpha_{ij}^{(+)}\} \subset \mathcal{H}_2$$

for any $i \in I$ and $j \in J$.

Since $\alpha_{ij}^{(+)} \subseteq \mathbb{T}_+ \cup \{+\infty\}$ and $\alpha_{ij}^{(-)} \subseteq \mathbb{T}_+ \cup \{0\}$, observe that the sets $\alpha_{i_1 j_1}^{(+)}$ and $\alpha_{i_2 j_2}^{(+)}$, as well as $\alpha_{i_1 j_1}^{(-)}$ and $\alpha_{i_2 j_2}^{(-)}$, can be unambiguously multiplied (by definition, the product of two sets consists of all possible products of an element of one set by an element of the other set) for any $i_1, i_2 \in I$ and $j_1, j_2 \in J$.

We now formulate one of the main results of our paper: a characterization of conic hemispaces in terms of their generating sets. We will immediately prove that any couple of conic hemispaces fulfills the given conditions. The proof that these conditions are also sufficient is going to occupy the remaining part of this section.

In the sequel, we write $I^1 + \dots + I^m = I$ if I^k for $k \in [m]$ and I are index sets such that $I^1 \cup \dots \cup I^m = I$ and I^1, \dots, I^m are pairwise disjoint.

Theorem 4.10. *The tropical cones $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ form a couple of conic hemispaces if and only if there exist subsets I, J of $[n]$ and pairs of subsets $(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)})$ of $\mathbb{T} \cup \{+\infty\}$ of the form (20) for $i \in I$ and $j \in J$, such that $I + J = [n]$,*

$$(22) \quad \alpha_{i_1 j_2}^{(+)} \alpha_{i_2 j_1}^{(+)} \cap \alpha_{i_1 j_1}^{(-)} \alpha_{i_2 j_2}^{(-)} = \emptyset \quad \text{and} \quad \alpha_{i_1 j_2}^{(-)} \alpha_{i_2 j_1}^{(-)} \cap \alpha_{i_1 j_1}^{(+)} \alpha_{i_2 j_2}^{(+)} = \emptyset$$

for any $i_1, i_2 \in I$ and $j_1, j_2 \in J$, and

$$(23) \quad \begin{aligned} \mathcal{H}_1 &= \text{span} \left(\left\{ e^i \oplus \lambda e^j \mid i \in I, j \in J, \lambda \in \alpha_{ij}^{(-)} \right\} \right), \\ \mathcal{H}_2 &= \text{span} \left(\left\{ e^i \oplus \lambda e^j \mid i \in I, j \in J, \lambda \in \alpha_{ij}^{(+)} \right\} \right). \end{aligned}$$

Proof of the “only if” part. Given a pair $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ of conic hemispaces, let I and J be the sets defined in (18), and $(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)})$ be the pairs of subsets of $\mathbb{T} \cup \{+\infty\}$ defined by (19) and (20).

By Theorem 4.2, both \mathcal{H}_1 and \mathcal{H}_2 are generated by unit vectors and combinations of two unit vectors. The distribution of unit vectors is given by I and J . Observe that (23) conforms to this distribution, since for any $i \in I$, e^i belongs to the generators of \mathcal{H}_1 as $0 \in \alpha_{ij}^{(-)}$, and for any $j \in J$, e^j belongs to the generators of \mathcal{H}_2 since $+\infty \in \alpha_{ij}^{(+)}$. Moreover, this obviously implies that no combination of e^{i_1} and e^{i_2} with $i_1, i_2 \in I$ (resp. of e^{j_1} and e^{j_2} with $j_1, j_2 \in J$) is necessary in (23) to generate \mathcal{H}_1 (resp. \mathcal{H}_2). For $i \in I$ and $j \in J$, the distribution of the combinations of e^i and e^j is given by (21). These observations yield (23).

It remains to prove (22). Assume that

$$\alpha_{i_1 j_2}^{(+)} \alpha_{i_2 j_1}^{(+)} \cap \alpha_{i_1 j_1}^{(-)} \alpha_{i_2 j_2}^{(-)} \neq \emptyset.$$

Then, there exist $\beta_{i_1 j_2} \in \alpha_{i_1 j_2}^{(+)}$, $\beta_{i_2 j_1} \in \alpha_{i_2 j_1}^{(+)}$, $\gamma_{i_1 j_1} \in \alpha_{i_1 j_1}^{(-)}$ and $\gamma_{i_2 j_2} \in \alpha_{i_2 j_2}^{(-)}$ such that $\beta_{i_1 j_2} \beta_{i_2 j_1} = \gamma_{i_1 j_1} \gamma_{i_2 j_2}$. For this to hold, the products $\beta_{i_1 j_2} \beta_{i_2 j_1}$ and $\gamma_{i_1 j_1} \gamma_{i_2 j_2}$ should be in \mathbb{T}_+ , and hence $\beta_{i_1 j_2}$, $\beta_{i_2 j_1}$, $\gamma_{i_1 j_1}$ and $\gamma_{i_2 j_2}$ should be in \mathbb{T}_+ . Then, we make the combination

$$z = e^{i_1} \oplus \beta_{i_1 j_2} e^{j_2} \oplus \lambda (e^{i_2} \oplus \beta_{i_2 j_1} e^{j_1}) \in \mathcal{H}_2,$$

where λ satisfies $\lambda \beta_{i_2 j_1} = \gamma_{i_1 j_1}$, hence also $\lambda \gamma_{i_2 j_2} = \beta_{i_1 j_2}$, and observe that

$$z = e^{i_1} \oplus \gamma_{i_1 j_1} e^{j_1} \oplus \lambda (e^{i_2} \oplus \gamma_{i_2 j_2} e^{j_2}) \in \mathcal{H}_1.$$

Thus $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \{0\}$, a contradiction. \square

Condition 22 will be called the **rank-one condition**, due to the following observation.

Corollary 4.11. *If $\alpha_{ij} \in \mathbb{T}_+$ for $i \in \{i_1, i_2\}$ and $j \in \{j_1, j_2\}$, then $\alpha_{i_1 j_1} \alpha_{i_2 j_2} = \alpha_{i_1 j_2} \alpha_{i_2 j_1}$. In particular, if $\alpha_{ij} \in \mathbb{T}_+$ for all $i \in I$ and $j \in J$, then the α -matrix has rank one.*

In the rest of this subsection, we assume that \mathcal{H}_1 is a tropical cone defined as in (23), where $I + J = [n]$ and the sets $\alpha_{ij}^{(-)}$, which are either of the form $\{\lambda \in \mathbb{T} \mid \lambda \leq \alpha_{ij}\}$ or $\{\lambda \in \mathbb{T} \mid \lambda < \alpha_{ij}\}$ with $\alpha_{ij} \in \mathbb{T} \cup \{+\infty\}$, are such that the pairs $(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)})$ satisfy the rank-one condition (22) if we define $\alpha_{ij}^{(+)} := (\mathbb{T} \cup \{+\infty\}) \setminus \alpha_{ij}^{(-)}$. With the objective of showing that any such cone is a conic hemispace, we first give a detailed description of the “thin structure” of the corresponding α -matrix that follows from the rank-one condition (22). This description can be also seen as one of our main results.

Proposition 4.12. *If we define*

$$\begin{aligned} J_i^< &:= \{j \in J \mid \alpha_{ij} \in \mathbb{T}_+ \text{ and } \alpha_{ij} \in \alpha_{ij}^{(+)}\}, \\ J_i^{\leq} &:= \{j \in J \mid \alpha_{ij} \in \mathbb{T}_+ \text{ and } \alpha_{ij} \in \alpha_{ij}^{(-)}\}, \\ J_i^0 &:= \{j \in J \mid \alpha_{ij} = 0\}, \\ J_i^\infty &:= \{j \in J \mid \alpha_{ij} = +\infty\}, \end{aligned}$$

for $i \in I$, by the rank-one condition (22) it follows that:

- (i) $J_i^{\leq} + J_i^< + J_i^\infty + J_i^0 = J$ for each $i \in I$;
- (ii) $J_{i_1}^\infty \subseteq J_{i_2}^\infty$ or $J_{i_2}^\infty \subseteq J_{i_1}^\infty$, and $J_{i_1}^0 \subseteq J_{i_2}^0$ or $J_{i_2}^0 \subseteq J_{i_1}^0$ for any $i_1, i_2 \in I$;
- (iii) If $(J_{i_1}^< \cup J_{i_1}^{\leq}) \cap (J_{i_2}^< \cup J_{i_2}^{\leq}) \neq \emptyset$, then $J_{i_1}^< \cup J_{i_1}^{\leq} = J_{i_2}^< \cup J_{i_2}^{\leq}$, $J_{i_1}^\infty = J_{i_2}^\infty$, $J_{i_1}^0 = J_{i_2}^0$;
- (iv) If $(J_{i_1}^< \cup J_{i_1}^{\leq}) \cap (J_{i_2}^< \cup J_{i_2}^{\leq}) \neq \emptyset$, then $J_{i_1}^< \subseteq J_{i_2}^<$ or $J_{i_2}^< \subseteq J_{i_1}^<$;
- (v) If $(J_{i_1}^< \cup J_{i_1}^{\leq}) \cap (J_{i_2}^< \cup J_{i_2}^{\leq}) \neq \emptyset$, then there exists $\lambda \in \mathbb{T}_+$ such that $\alpha_{i_1 j} = \lambda \alpha_{i_2 j}$ for all $j \in J_{i_1}^< \cup J_{i_1}^{\leq} = J_{i_2}^< \cup J_{i_2}^{\leq}$.

Proof. In this proof, we will use F , $\geq F$ and $\leq F$ to represent an entry of a matrix which belongs to \mathbb{T}_+ , $\mathbb{T}_+ \cup \{+\infty\}$ and $\mathbb{T}_+ \cup \{0\} = \mathbb{T}$, respectively.

- (i) This property readily follows from the definition of the sets $J_i^<$, J_i^{\leq} , J_i^0 , and J_i^∞ .
- (ii) If these conditions are violated, then the α -matrix has one of the following 2×2 minors

$$\begin{pmatrix} 0 & \geq F \\ \geq F & 0 \end{pmatrix}, \quad \begin{pmatrix} +\infty & \leq F \\ \leq F & +\infty \end{pmatrix},$$

violating (22).

- (iii) If this condition is violated, then the α -matrix has one of the following 2×2 minors

$$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}, \quad \begin{pmatrix} F & F \\ +\infty & F \end{pmatrix}, \quad \begin{pmatrix} +\infty & F \\ 0 & F \end{pmatrix},$$

violating (22). More precisely, one of the first two minors will appear when $(J_{i_1}^< \cup J_{i_1}^{\leq}) \cap (J_{i_2}^< \cup J_{i_2}^{\leq}) \neq \emptyset$ but $(J_{i_1}^< \cup J_{i_1}^{\leq}) \neq (J_{i_2}^< \cup J_{i_2}^{\leq})$. The third one will appear if $(J_{i_1}^< \cup J_{i_1}^{\leq}) = (J_{i_2}^< \cup J_{i_2}^{\leq}) \neq \emptyset$ but $J_{i_1}^\infty \neq J_{i_2}^\infty$ (equivalently, $J_{i_1}^0 \neq J_{i_2}^0$).

- (iv) If $J_{i_1}^< \subseteq J_{i_2}^<$ and $J_{i_2}^< \subseteq J_{i_1}^<$ do not hold for some i_1, i_2 , then there exist j_1 and j_2 such that $\alpha_{i_1 j_1} \in \alpha_{i_1 j_1}^{(+)}$, $\alpha_{i_2 j_2} \in \alpha_{i_2 j_2}^{(+)}$, $\alpha_{i_1 j_2} \in \alpha_{i_1 j_2}^{(-)}$, $\alpha_{i_2 j_1} \in \alpha_{i_2 j_1}^{(-)}$, and $\alpha_{i_1 j_1}, \alpha_{i_1 j_2}, \alpha_{i_2 j_1}, \alpha_{i_2 j_2} \in \mathbb{T}_+$. However, this contradicts the rank-one condition (22), since $\alpha_{i_1 j_1} \alpha_{i_2 j_2} = \alpha_{i_1 j_2} \alpha_{i_2 j_1}$ by Corollary 4.11.
- (v) This property follows from Corollary 4.11 and Property (iii). \square

Remark 4.13. *Regarding Property (ii) of Proposition 4.12, observe that condition “ $J_{i_1}^\infty \subseteq J_{i_2}^\infty$ or $J_{i_2}^\infty \subseteq J_{i_1}^\infty$ ” can be equivalently formulated as “ $J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^0 \subseteq J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^0$ or $J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^0 \subseteq J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^0$ ” for any $i_1, i_2 \in I$. Similarly, condition “ $J_{i_1}^0 \subseteq J_{i_2}^0$ or $J_{i_2}^0 \subseteq J_{i_1}^0$ ” can be equivalently formulated as “ $J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^\infty \subseteq J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^\infty$ or $J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^\infty \subseteq J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^\infty$ ” for any $i_1, i_2 \in I$.*

Consider the equivalence relation on I defined by

$$i_1 \sim i_2 \iff J_{i_1}^\infty = J_{i_2}^\infty \text{ and } J_{i_1}^0 = J_{i_2}^0.$$

By Proposition 4.12 part (ii), the relation

$$i_1 \preceq i_2 \iff \begin{cases} J_{i_2}^\infty \subseteq J_{i_1}^\infty \text{ or} \\ J_{i_1}^\infty = J_{i_2}^\infty \text{ and } J_{i_1}^0 \subseteq J_{i_2}^0 \end{cases}$$

defines a total order on I , which induces a total order (also denoted by \preceq) on the equivalence classes associated with \sim . Assume that I^1, \dots, I^p are these equivalence classes and that $I^1 \preceq I^2 \preceq \dots \preceq I^p$.

By definition, note that there exist subsets L^1, \dots, L^p , K^1, \dots, K^p , and J^1, \dots, J^p of J , such that $J_i^0 = L^r$, $J_i^\infty = K^r$ and $J_i^< \cup J_i^{\leq} = J^r$ for $i \in I^r$. Thus, by Proposition 4.12 part (i), it follows that

$$J^r + K^r + L^r = J$$

for $r \in [p]$, and from part (iii) we conclude that the sets J^1, \dots, J^p are pairwise disjoint. Moreover, for $r \in [2, p]$ we have

$$(24) \quad J^r \cup K^r \subseteq K^{r-1},$$

or equivalently

$$J^{r-1} \cup L^{r-1} \subseteq L^r.$$

Indeed, if $i_1 \in I^{r-1}$ and $i_2 \in I^r$, using Remark 4.13 we conclude that either $J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^\infty \subseteq J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^\infty$ or $J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^\infty \subseteq J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^\infty$. Since $J^{r-1} = J_{i_1}^< \cup J_{i_1}^{\leq}$ and $J^r = J_{i_2}^< \cup J_{i_2}^{\leq}$ are disjoint and by Proposition 4.12 part (ii), it follows that either $J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^\infty \subseteq J_{i_1}^\infty$ or $J_{i_1}^< \cup J_{i_1}^{\leq} \cup J_{i_1}^\infty \subseteq J_{i_2}^\infty$. In the former case, we have $J^r \cup K^r = J_{i_2}^< \cup J_{i_2}^{\leq} \cup J_{i_2}^\infty \subseteq J_{i_1}^\infty = K^{r-1}$. In the latter case, as $i_1 \preceq i_2$, we have $J_{i_2}^\infty \subseteq J_{i_1}^\infty$ and so $K^{r-1} = J_{i_1}^\infty = J_{i_2}^\infty = K^r$ and $J^{r-1} = J_{i_1}^< \cup J_{i_1}^{\leq} = \emptyset$. Thus, $L^{r-1} = J_{i_1}^0 \subseteq J_{i_2}^0 = L^r$ because $i_1 \preceq i_2$, which implies $J^r \cup K^r = J \setminus L^r \subseteq J \setminus L^{r-1} = J^{r-1} \cup K^{r-1} = K^{r-1}$.

Finally, note that by Proposition 4.12 part (iv), we have

$$(25) \quad J_{i_1}^< \subseteq J_{i_2}^< \text{ or } J_{i_2}^< \subseteq J_{i_1}^<$$

for all $i_1, i_2 \in I^r$ and $r \in [p]$.

Observe that \mathcal{H}_1 is also generated by the set

$$\bigcup_{i \in I} \left(\{e^i\} \cup \{e^i \oplus \alpha_{ij} e^j \mid j \in J_i^{\leq}\} \cup \{e^i \oplus \lambda e^j \mid j \in J_i^<, \lambda < \alpha_{ij}\} \cup \{e^i \oplus \lambda e^j \mid j \in J_i^\infty, \lambda \in \mathbb{R}\} \right),$$

since any vector of the form $e^i \oplus \lambda e^j$, where $j \in J_i^<$ and $\lambda < \alpha_{ij}$, can be expressed as a (tropical linear) combination of $e^i \oplus \alpha_{ij} e^j$ and e^i . Moreover, defining

$$(26) \quad \begin{aligned} \mathcal{C}_i &:= \text{span} \left(\{e^i\} \cup \{e^i \oplus \alpha_{ij} e^j \mid j \in J_i^{\leq}\} \cup \{e^i \oplus \lambda e^j \mid j \in J_i^<, \lambda < \alpha_{ij}\} \right), \\ \mathcal{D}_i &:= \text{span} \left(\{e^i\} \cup \{e^i \oplus \lambda e^j \mid j \in J_i^\infty, \lambda \in \mathbb{T}_+\} \right), \end{aligned}$$

for $i \in I$, we have $\mathcal{H}_1 = \bigoplus_{i \in I} (\mathcal{C}_i \oplus \mathcal{D}_i)$.

Lemma 4.14. *There exist $\beta_h \in \mathbb{T}_+$, for $h \in I$, and $\gamma_j \in \mathbb{T}_+$, for $j \in \bigcup_{i \in I} (J_i^{\leq} \cup J_i^<)$, such that for each $i \in I$, the set of non-null vectors of the tropical cone \mathcal{C}_i is the set of vectors satisfying*

$$(27) \quad \begin{cases} \gamma_j x_j \leq \beta_i x_i & \text{for all } j \in J_i^{\leq} \\ \gamma_j x_j < \beta_i x_i & \text{for all } j \in J_i^< \\ x_j = 0 & \text{for all } j \in J_i^0 \cup J_i^\infty \cup (I \setminus \{i\}) \end{cases}$$

Proof. Proposition 4.12 part (v) implies that there exist $\beta_i, \gamma_j \in \mathbb{T}_+$ such that $\alpha_{ij} = \gamma_j^{-1} \beta_i$ for all $\alpha_{ij} \in \mathbb{T}_+$. Thus, the tropical cone \mathcal{C}_i can be equivalently defined by

$$\mathcal{C}_i = \text{span} \left(\{e^i\} \cup \{\gamma_j e^i \oplus \beta_i e^j \mid j \in J_i^{\leq}\} \cup \{\gamma_j e^i \oplus \lambda \beta_i e^j \mid j \in J_i^<, \lambda < 1\} \right).$$

Next, any $x \in \mathcal{C}_i$ can be written as a combination of vectors in the cones

$$\begin{aligned} \mathcal{C}_{ij}^{\leq} &:= \text{span} \left(\{e^i\} \cup \{\gamma_j e^i \oplus \beta_i e^j \mid j \in J_i^{\leq}\} \right), \\ \mathcal{C}_{ij}^< &:= \text{span} \left(\{e^i\} \cup \{\gamma_j e^i \oplus \lambda \beta_i e^j \mid j \in J_i^<, \lambda < 1\} \right), \end{aligned}$$

with the same coefficient x_i at e^i . The generators of \mathcal{C}_{ij}^{\leq} and $\mathcal{C}_{ij}^<$ satisfy the first and second conditions of (27) respectively, hence x also satisfies all these conditions. Conversely, each non-null vector x satisfying (27) can be written (using similar ideas to those in the proof of Lemma 4.1) as a combination of the generators of \mathcal{C}_{ij}^{\leq} and $\mathcal{C}_{ij}^<$, and so it belongs to \mathcal{C}_i . \square

Later we will show that certain Minkowski sums of the tropical cones \mathcal{C}_i are conic hemispaces. To this end, note that we have $\mathcal{C}_i = \{x \in \mathbb{T}^n \mid x_j = 0 \text{ for } j \neq i\}$ if $J_i^< \cup J_i^{\leq} = \emptyset$, and so

$$(28) \quad \bigoplus_{i \in \tilde{I}} \mathcal{C}_i = \left\{ x \in \mathbb{T}^n \mid x_j = 0 \text{ for all } j \notin \tilde{I} \right\}$$

when $J_i^< \cup J_i^{\leq} = \emptyset$ for all $\tilde{I} \subseteq I$ and $i \in \tilde{I}$. Evidently, any set given by (28) is a conic hemispace.

Since $\mathcal{H}_1 = \bigoplus_{i \in I} (\mathcal{C}_i \oplus \mathcal{D}_i)$, observe that the (tropical) null vector is the only vector x in \mathcal{H}_1 satisfying $x_i = 0$ for all $i \in I$.

Theorem 4.15. *Given $x \in \mathbb{T}^n$, if $x_i \neq 0$ for some $i \in I$, let $h := \min\{r \in [p] \mid x_t \neq 0 \text{ for some } t \in I^r\}$ and $\hat{x} \in \mathbb{T}^n$ be the vector defined by $\hat{x}_k := 0$ if $k \in (\bigcup_{r > h} I^r) \cup K^h$ and $\hat{x}_k := x_k$ otherwise. Then, $x \in \mathcal{H}_1$ if and only if $\hat{x} \in \bigoplus_{i \in I^h} \mathcal{C}_i$.*

Proof. The “if” part: Observe that $\hat{x} \in \bigoplus_{i \in I^h} \mathcal{C}_i \subseteq \mathcal{H}_1$, $e^i \in \mathcal{H}_1$ for all $i \in I$ and $e^i \oplus \lambda e^j \in \mathcal{H}_1$ for all $i \in I^h$, $j \in K^h$ and $\lambda \in \mathbb{R}$. Since $x_t \neq 0$ for some $t \in I^h$, x can be written as a combination of these vectors, so it also belongs to \mathcal{H}_1 .

The “only if” part: Let $x \in \mathcal{H}_1$. As $\mathcal{H}_1 = \bigoplus_{i \in I} (\mathcal{C}_i \oplus \mathcal{D}_i)$, we have $x = \bigoplus_{i \in I} (y^i \oplus z^i)$ for some $y^i \in \mathcal{C}_i$ and $z^i \in \mathcal{D}_i$. Note that $y^i \oplus z^i = 0$ for $i \in I^r$ with $r < h$ since $y_i^i \oplus z_i^i = x_i = 0$ for such vectors. So $x = \bigoplus_{i \in \bigcup_{r \geq h} I^r} (y^i \oplus z^i)$.

We will show that y^i can be chosen so that $\hat{x} = \bigoplus_{i \in I^h} y^i \in \bigoplus_{i \in I^h} \mathcal{C}_i$. For this, observe that for all $i \in I^h$, since $e^i \in \mathcal{C}_i$, we can assume $x_i = \hat{x}_i = y_i^i$, adding $x_i e^i$ to y^i if necessary. This fixes our choice of y^i . Then by (24), for $r > h$ we have $J^r \cup K^r \subseteq K^h$, or equivalently, $J^h \cup L^h \subseteq L^r$. Recalling that $L^r = J_i^0$ for all $i \in I^r$, we see that the subvector of x or \hat{x} restricted to $I^h \cup J^h \cup L^h$, is equal to the corresponding subvector of $\bigoplus_{i \in I^h} y^i$, because y^i with $i \in I^r$ and $r > h$ do not contribute having $\text{supp}(y^i) \subseteq I^r \cup K^r \cup J^r = I^r \cup (J \setminus L^r)$. Moreover, observe that z^i for $i \in I^r$ and $r \geq h$ do not contribute either, due to the fact that $\text{supp}(z^i) \subseteq K^r \cup \{i\} \subseteq K^h \cup \{i\}$ and $x_i = \hat{x}_i = y_i^i$. Since the subvectors of \hat{x} and $\bigoplus_{i \in I^h} y^i$ restricted to the complement of $I^h \cup J^h \cup L^h$ are 0, the claim follows. \square

We now describe $\bigoplus_{i \in I^r} \mathcal{C}_i$ as set of vectors lying in a halfspace (29) and satisfying a constraint (30).

Lemma 4.16. *If $J^r \neq \emptyset$, then the non-null elements of the tropical cone $\bigoplus_{i \in I^r} \mathcal{C}_i$ are the vectors $x \in \mathbb{T}^n$ that satisfy $x_i \neq 0$ for some $i \in I^r$,*

$$(29) \quad \bigoplus_{j \in J^r} \gamma_j x_j \leq \bigoplus_{i \in I^r} \beta_i x_i \text{ and } x_j = 0 \text{ for } j \notin I^r \cup J^r,$$

and, in addition,

$$(30) \quad \gamma_j x_j = \bigoplus_{i \in I^r} \beta_i x_i \implies \exists k \in I^r \text{ such that } \gamma_j x_j = \beta_k x_k \text{ and } j \in J_k^{\leq}.$$

Proof. Assume first that the conditions are satisfied for $x \in \mathbb{T}^n$. Given $j \in J^r$, if $\gamma_j x_j = \bigoplus_{i \in I^r} \beta_i x_i$, let $k \in I^r$ be such that $\beta_k x_k = \bigoplus_{i \in I^r} \beta_i x_i$ and $j \in J_k^{\leq}$. Then, the vector $y^{kj} := e^k \oplus x_j x_k^{-1} e^j$ belongs to \mathcal{C}_k because $j \in J_k^{\leq}$ and $x_j x_k^{-1} = \beta_k \gamma_j^{-1} = \alpha_{kj}$. Given $j \in J^r$ such that $\gamma_j x_j < \bigoplus_{i \in I^r} \beta_i x_i$, let k be any element of I^r such that $\beta_k x_k$ attains the maximum in $\bigoplus_{i \in I^r} \beta_i x_i$. The vector $y^{kj} := e^k \oplus x_j x_k^{-1} e^j$ again belongs to \mathcal{C}_k , because $j \in J_k^{\leq} \cup J_k^<$ and $x_j x_k^{-1} < \beta_k \gamma_j^{-1} = \alpha_{kj}$ so $x_j x_k^{-1} < \alpha_{kj}$. Since $e^i \in \mathcal{C}_i$ for all $i \in I^r$, it readily follows that $x \in \bigoplus_{i \in I^r} \mathcal{C}_i$ as a sum of $x_i e^i$ for $i \in I^r$ and $x_k y^{kj} = x_k e^k \oplus x_j e^j$ over all y^{kj} considered above.

Assume now that $x \in \bigoplus_{i \in I^r} \mathcal{C}_i$ is non-null. Using (27) we observe that each vector y in \mathcal{C}_i for $i \in I^r$ satisfies $\bigoplus_{j \in J^r} \gamma_j y_j \leq \beta_i y_i$ and $y_h = 0$ for all $h \notin I^r \cup J^r$, hence it lies in the halfspace (29), and so the same holds for x . Condition (29) implies that $x_i \neq 0$ for some $i \in I^r$. Represent $x = \bigoplus_{i \in I^r} y^i$ where $y^i \in \mathcal{C}_i$. If $\gamma_j x_j = \bigoplus_{i \in I^r} \beta_i x_i$, let $k \in I^r$ be such that $x_j = y_j^k$. Since $y^k \in \mathcal{C}_k$, we have $\gamma_j y_j^k \leq \beta_k y_k^k$, and it follows that $\gamma_j x_j = \gamma_j y_j^k \leq \beta_k y_k^k \leq \beta_k x_k \leq \bigoplus_{i \in I^r} \beta_i x_i$. All these inequalities turn into the equalities, so we have $\gamma_j y_j^k = \beta_k y_k^k$ with $y^k \in \mathcal{C}_k$, and hence $j \in J_k^{\leq}$ by (27). This shows that the conditions of the lemma are also necessary. \square

Proposition 4.17. *For each $r \in [p]$ the tropical cone $\bigoplus_{i \in I^r} \mathcal{C}_i$ is a conic hemispace.*

Proof. The case when $J^r = \emptyset$ was treated in (28), so we can assume $J^r \neq \emptyset$. We have shown that the nontrivial elements of $\bigoplus_{i \in I^r} \mathcal{C}_i$ are precisely the elements of \mathbb{T}^n that satisfy (29) and (30). In the rest of the proof, we assume that the complement of $I^r \cup J^r$ is empty, or equivalently, we will show that $\bigoplus_{i \in I^r} \mathcal{C}_i$ is a conic hemispace in the plane $\{x_i = 0 \mid i \notin I^r \cup J^r\}$, from which it follows that $\bigoplus_{i \in I^r} \mathcal{C}_i$ is a conic hemispace in \mathbb{T}^n . (For this, verify that the complement of a cone lying in $\{x_i = 0 \mid i \in \tilde{I}\}$, for \tilde{I} a subset of $[n]$, is a cone, if the restriction of that complement to $\{x_i = 0 \mid i \in \tilde{I}\}$ is a cone.) Thus, we assume $I^r \cup J^r = [n]$.

Let us build a “reflection” of $\bigoplus_{i \in I^r} \mathcal{C}_i$, swapping the roles of I^r and J^r , and the roles of $J_k^<$ and $J_k^<$ in (29) and (30). Namely, we define it as the set $\tilde{\mathcal{C}}$ containing all the vectors $x \in \mathbb{T}^n$ that satisfy

$$(31) \quad \bigoplus_{i \in I^r} \beta_i x_i \leq \bigoplus_{j \in J^r} \gamma_j x_j$$

and

$$(32) \quad \beta_i x_i = \bigoplus_{j \in J^r} \gamma_j x_j \implies \exists k \in J^r \text{ such that } \gamma_k x_k = \beta_i x_i \text{ and } k \in J_i^<.$$

We need to show that $\tilde{\mathcal{C}}$ is a tropical cone. Evidently, $x \in \tilde{\mathcal{C}}$ implies $\lambda x \in \tilde{\mathcal{C}}$ for all $\lambda \in \mathbb{R}$. If $x, y \in \tilde{\mathcal{C}}$ and $z = x \oplus y$ satisfies (31) with strict inequality, then $z \in \tilde{\mathcal{C}}$. If not, let i be such that $\beta_i z_i = \bigoplus_{j \in J^r} \gamma_j z_j$, and assume $z_i = x_i$. It follows that $\beta_i x_i = \bigoplus_{j \in J^r} \gamma_j x_j$, and then there exists $k \in J^r$ such that $\gamma_k x_k = \beta_i x_i$ and $k \in J_i^<$. Further observe that $\gamma_k z_k \geq \gamma_k x_k = \beta_i x_i = \beta_i z_i = \bigoplus_{j \in J^r} \gamma_j z_j \geq \gamma_k z_k$, and so $\gamma_k z_k = \beta_i z_i$, showing that z satisfies (32) and is in $\tilde{\mathcal{C}}$.

Moreover, it can be shown that $\tilde{\mathcal{C}} = \bigoplus_{j \in J^r} \tilde{\mathcal{C}}_j$, where $\tilde{\mathcal{C}}_j$ are defined as the “reflection” of \mathcal{C}_i , i.e., tropical cones whose non-null vectors satisfy

$$\begin{cases} \beta_i x_i \leq \gamma_j x_j \text{ for all } i \text{ such that } j \in J_i^< \\ \beta_i x_i < \gamma_j x_j \text{ for all } i \text{ such that } j \in J_i^< \\ x_i = 0 \text{ for all } i \in J^r \setminus \{j\} \end{cases}$$

The proof of $\tilde{\mathcal{C}} = \bigoplus_{j \in J^r} \tilde{\mathcal{C}}_j$ is based on the arguments of Lemmas 4.14 and 4.16. (This observation is just a valuable remark not used in the current proof.)

We now show that $\tilde{\mathcal{C}}$ is the complement of $\bigoplus_{i \in I^r} \mathcal{C}_i$, so they form a couple of conic hemispaces. Building the complement of $\bigoplus_{i \in I^r} \mathcal{C}_i$ by negating (29) and (30), we see that it consists of two branches: vectors x satisfying

$$\bigoplus_{i \in I^r} \beta_i x_i < \bigoplus_{j \in J^r} \gamma_j x_j,$$

and those satisfying

$$\bigoplus_{i \in I^r} \beta_i x_i = \bigoplus_{j \in J^r} \gamma_j x_j$$

and

$$\exists k \in J^r \text{ such that } \gamma_k x_k = \bigoplus_{i \in I^r} \beta_i x_i, \text{ and } k \in J_h^< \text{ whenever } \beta_h x_h = \gamma_k x_k.$$

It can be verified that both branches belong to the “reflection” $\tilde{\mathcal{C}}$ as defined by (31) and (32).

We are now left to show that $\bigoplus_{i \in I^r} \mathcal{C}_i$ and its “reflection” $\tilde{\mathcal{C}}$ do not contain any common non-null vector. We will use (25), i.e., the fact that for each $i_1, i_2 \in I^r$ either $J_{i_1}^< \subseteq J_{i_2}^<$ or $J_{i_2}^< \subseteq J_{i_1}^<$. This property means that the sets $J_i^<$ and $J_i^< = J \setminus J_i^<$ are nested, hence the elements of I^r and J^r can be assumed to be ordered so that

$$i_1 \leq i_2 \Leftrightarrow J_{i_2}^< \subseteq J_{i_1}^<$$

and the following properties are satisfied:

$$(33) \quad \begin{aligned} j_1 \in J_{i_1}^<, j_2 \in J_{i_1}^< &\implies j_1 < j_2, \\ j_1 \in J_{i_1}^<, j_1 \in J_{i_2}^< &\implies i_2 < i_1. \end{aligned}$$

Assume now $x \in (\bigoplus_{i \in I^r} \mathcal{C}_i) \cap \tilde{\mathcal{C}}$ but $x \neq 0$. Then, we necessarily have $\bigoplus_{i \in I^r} \beta_i x_i = \bigoplus_{j \in J^r} \gamma_j x_j \neq 0$. Let $i_1 \in I^r$ be such that $\beta_{i_1} x_{i_1} = \bigoplus_{j \in J^r} \gamma_j x_j$. Since $x \in \tilde{\mathcal{C}}$, there exists $j_1 \in J_{i_1}^<$ such that $\bigoplus_{j \in J^r} \gamma_j x_j = \gamma_{j_1} x_{j_1}$. As $x \in \bigoplus_{i \in I^r} \mathcal{C}_i$, there exists $i_2 \in I^r$ such that $\beta_{i_2} x_{i_2} = \bigoplus_{i \in I^r} \beta_i x_i = \gamma_{j_1} x_{j_1}$ and $j_1 \in J_{i_2}^<$, and so $i_2 < i_1$ by (33). Again, using the fact that $x \in \tilde{\mathcal{C}}$ and $\beta_{i_2} x_{i_2} = \bigoplus_{j \in J^r} \gamma_j x_j$, we conclude that there exists $j_2 \in J_{i_2}^<$ such that $\bigoplus_{j \in J^r} \gamma_j x_j = \gamma_{j_2} x_{j_2}$, and so $j_1 < j_2$ by (33). Repeating this argument again and again we obtain infinite sequences $i_1 > i_2 > i_3 > \dots$ and $j_1 < j_2 < j_3 < \dots$, which is impossible. Hence, $\bigoplus_{i \in I^r} \mathcal{C}_i$ and $\tilde{\mathcal{C}}$ form a couple of conic hemispaces. \square

Proof of the “if” part of Theorem 4.10. We next show that the tropical cone \mathcal{H}_1 defined in (23) is a conic hemispace if the rank-one condition (22) is satisfied. Since the generators of \mathcal{H}_2 in (23) are precisely those unit vectors and combinations of two unit vectors not belonging to \mathcal{H}_1 (except the evidently redundant ones), from Theorem 4.2 we conclude that \mathcal{H}_1 and \mathcal{H}_2 form a couple of conic hemispaces.

Let $\mathcal{C}_i \subset \mathbb{T}^n$, for $i \in I$, be defined by (26) (see also (27), a working equivalent definition, and Lemma 4.16 for an equivalent definition of $\bigoplus_{i \in I^r} \mathcal{C}_i$). Let the operator $x \mapsto \hat{x}$ be defined as in Theorem 4.15.

Given $x \in \mathcal{CH}_1$ and $\lambda \in \mathbb{T}_+$ (we assume $x_i \neq 0$ for some $i \in I$, otherwise $\lambda x \in \mathcal{CH}_1$ is immediate), let $h := \min\{r \in [p] \mid x_t \neq 0 \text{ for some } t \in I^r\}$. Then, $\hat{x} \notin \bigoplus_{i \in I^h} \mathcal{C}_i$ by Theorem 4.15 because $x \in \mathcal{CH}_1$. Note that for $y := \lambda x$ we have $\min\{r \in [p] \mid y_t \neq 0 \text{ for some } t \in I^r\} = h$ and $\hat{y} = \lambda \hat{x}$. By Theorem 4.15 it follows that $y \in \mathcal{CH}_1$ because $\hat{y} \in \bigoplus_{i \in I^h} \mathcal{C}_i$.

Let now $x, y \in \mathcal{CH}_1$ (which in particular means $x \neq 0$ and $y \neq 0$) and define $z := x \oplus y$.

Assume first that $x_i = y_i = 0$ for all $i \in I$. Then, $z_i = 0$ for all $i \in I$, and as $z \neq 0$, we conclude $z \in \mathcal{CH}_1$.

In the second place, assume $x_i \neq 0$ for some $i \in I$ but $y_t = 0$ for all $t \in I$. Then, note that $\hat{z} = \hat{x} \oplus w$ for some vector w which satisfies $\text{supp}(w) \cap I = \emptyset$. Let $h := \min\{r \in [p] \mid x_t \neq 0 \text{ for some } t \in I^r\}$, so $\hat{x} \notin \bigoplus_{i \in I^h} \mathcal{C}_i$ by Theorem 4.15. Since $\hat{z} = \hat{x} \oplus w$ and $\text{supp}(w) \cap I^h = \emptyset$, from Lemma 4.16 it follows that $\hat{z} \notin \bigoplus_{i \in I^h} \mathcal{C}_i$, and so $z \in \mathcal{CH}_1$ by Theorem 4.15.

Finally, assume $x_i \neq 0$ and $y_t \neq 0$ for some $i, t \in I$. Let $h := \min\{r \in [p] \mid x_t \neq 0 \text{ for some } t \in I^r\}$ and $k := \min\{r \in [p] \mid y_t \neq 0 \text{ for some } t \in I^r\}$. We first consider the case $h \neq k$, and so without loss of generality we may assume $h < k$. Then, as above, we conclude that $z \in \mathcal{CH}_1$ because $\hat{z} = \hat{x} \oplus w$ for some vector w satisfying $\text{supp}(w) \cap I^h = \emptyset$. Suppose now $h = k$. Then, $\min\{r \in [p] \mid z_t \neq 0 \text{ for some } t \in I^r\} = h$ and $\hat{z} = \hat{x} \oplus \hat{y}$. From $\hat{x} \notin \bigoplus_{i \in I^h} \mathcal{C}_i$ and $\hat{y} \notin \bigoplus_{i \in I^h} \mathcal{C}_i$, it follows that $\hat{z} \notin \bigoplus_{i \in I^h} \mathcal{C}_i$, because $\bigoplus_{i \in I^h} \mathcal{C}_i$ is a conic hemispace by Proposition 4.17. Thus, again by Theorem 4.15, we have $z \in \mathcal{CH}_1$. \square

Example 4.18. Assume that

$$\mathcal{H}_1 = \text{span}(\{e^1\} \cup \{e^1 \oplus e^3\} \cup \{e^1 \oplus \delta e^4 \mid \delta \in \mathbb{T}\} \cup \{e^2\} \cup \{e^2 \oplus e^4\})$$

and

$$\mathcal{H}_2 = \text{span}(\{e^3\} \cup \{e^3 \oplus \alpha e^1 \mid \alpha < 1\} \cup \{e^3 \oplus \beta e^2 \mid \beta \in \mathbb{T}\} \cup \{e^4\} \cup \{e^4 \oplus \gamma e^2 \mid \gamma < 1\}).$$

In the notation of Theorem 4.10, one has $I = \{1, 2\}, J = \{3, 4\}$, $(\alpha_{13}^{(-)}, \alpha_{13}^{(+)}) = (\{\lambda \mid \lambda \leq 1\}, \{\lambda \mid \lambda > 1\})$, $(\alpha_{14}^{(-)}, \alpha_{14}^{(+)}) = (\mathbb{T}, \{+\infty\})$, $(\alpha_{23}^{(-)}, \alpha_{23}^{(+)}) = (\{0\}, \mathbb{T}_+ \cup \{+\infty\})$ and $(\alpha_{24}^{(-)}, \alpha_{24}^{(+)}) = (\{\lambda \mid \lambda \leq 1\}, \{\lambda \mid \lambda > 1\})$.

We first show that $\mathcal{H}_1 \cap \mathcal{H}_2 = \{0\}$. Assume $x \in \mathcal{H}_1 \cap \mathcal{H}_2$. Note that we can always express x as a tropical linear combination of the generators of \mathcal{H}_1 containing at most one vector of the form $e^1 \oplus \delta e^4$. The same observation holds for the generators of \mathcal{H}_2 and vectors of the form $e^3 \oplus \alpha e^1$, $e^3 \oplus \beta e^2$ and $e^4 \oplus \gamma e^2$. Thus, we have

$$x = \mu_1 e^1 \oplus \mu_2 (e^1 \oplus e^3) \oplus \mu_3 (e^1 \oplus \delta e^4) \oplus \mu_4 e^2 \oplus \mu_5 (e^2 \oplus e^4)$$

for some $\mu_1, \mu_2, \mu_3, \mu_4, \mu_5 \in \mathbb{T}$ since $x \in \mathcal{H}_1$, and

$$x = \nu_1 e^3 \oplus \nu_2 (e^3 \oplus \alpha e^1) \oplus \nu_3 (e^3 \oplus \beta e^2) \oplus \nu_4 e^4 \oplus \nu_5 (e^4 \oplus \gamma e^2)$$

for some $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5 \in \mathbb{T}$ since $x \in \mathcal{H}_2$.

Writing the equality on components in these expressions gives:

$$\begin{aligned} \mu_1 \oplus \mu_2 \oplus \mu_3 &= \alpha \nu_2, \\ \mu_4 \oplus \mu_5 &= \nu_3 \beta \oplus \nu_5 \gamma, \\ \mu_2 &= \nu_1 \oplus \nu_2 \oplus \nu_3, \\ \mu_3 \delta \oplus \mu_5 &= \nu_4 \oplus \nu_5. \end{aligned} \tag{34}$$

From the first and third equalities in (34) it follows that

$$\mu_2 \leq \mu_1 \oplus \mu_2 \oplus \mu_3 = \alpha \nu_2 \leq \alpha (\nu_1 \oplus \nu_2 \oplus \nu_3) = \alpha \mu_2,$$

which, due to $\alpha < 1$, implies $\mu_1 = \mu_2 = \mu_3 = \nu_1 = \nu_2 = \nu_3 = 0$. Then, from the second and fourth equalities in (34) it follows that

$$\mu_5 \leq \mu_4 \oplus \mu_5 = \nu_5 \gamma \leq (\nu_4 \oplus \nu_5) \gamma = \mu_5 \gamma,$$

which, due to $\gamma < 1$, implies $\mu_4 = \mu_5 = \nu_4 = \nu_5 = 0$.

To show that $\mathcal{H}_1 \cup \mathcal{H}_2 = \mathbb{T}^4$, let $x \in \mathbb{T}^4$. It is convenient to consider different cases.

If $x_1 = x_3 = 0$, we have $x = x_4(e^2 \oplus e^4) \oplus x_2e^2 \in \mathcal{H}_1$ when $x_2 \geq x_4$, and defining $\gamma = x_4^{-1}x_2$ we have $x = x_4(e^4 \oplus \gamma e^2) \in \mathcal{H}_2$ when $x_2 < x_4$.

When $x_1 = 0$ and $x_3 \neq 0$, defining $\beta = x_3^{-1}x_2$ we have $x = x_4e^4 \oplus x_3(e^3 \oplus \beta e^2) \in \mathcal{H}_2$.

When $x_1 \neq 0$ and $x_3 = 0$, defining $\delta = x_1^{-1}x_4$ we have $x = x_2e^2 \oplus x_1(e^1 \oplus \delta e^4) \in \mathcal{H}_1$.

If $x_1 \neq 0$ and $x_3 \neq 0$, defining $\delta = x_1^{-1}x_4$ we have $x = x_1e^1 \oplus x_2e^2 \oplus x_3(e^1 \oplus e^3) \oplus x_1(e^1 \oplus \delta e^4) \in \mathcal{H}_1$ when $x_1 \geq x_3$, and defining $\beta = x_3^{-1}x_2$ and $\alpha = x_3^{-1}x_1$ we have $x = x_3e^3 \oplus x_4e^4 \oplus x_3(e^3 \oplus \beta e^2) \oplus x_3(e^3 \oplus \alpha e^1) \in \mathcal{H}_2$ when $x_1 < x_3$.

4.3. Closed hemispaces, closed halfspaces and general hemispaces. We now consider the case of closed conic hemispaces, and show that these are precisely the closed homogeneous halfspaces, i.e., tropical cones of the form

$$(35) \quad \left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in J} \gamma_j x_j \leq \bigoplus_{i \in I} \beta_i x_i \text{ and } x_i = 0 \text{ for all } i \in L \right\},$$

where I, J and L (with I and J , or L , possibly empty) are pairwise disjoint subsets of $[n]$.

Theorem 4.19 (Briec and Horvath [2]). *Closed conic hemispaces = closed homogeneous halfspaces.*

Proof. Closed homogeneous halfspaces are closed conic hemispaces, since the complement of (35) is given by

$$\left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in J} \gamma_j x_j > \bigoplus_{i \in I} \beta_i x_i \text{ or } x_i \neq 0 \text{ for some } i \in L \right\},$$

and so it is a tropical cone.

Conversely, if a conic hemispace is closed, then in (19) we have $\alpha_{ij} \in \mathbb{T}$ for all $i \in I$ and $j \in J$, and in (20) the definition of the pairs $(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)})$ specializes to

$$(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)}) = \begin{cases} (\{\lambda \mid \lambda \leq \alpha_{ij}\}, \{\lambda \mid \lambda > \alpha_{ij}\}) & \text{if } \alpha_{ij} \in \mathbb{T}_+, \\ (\{\alpha_{ij}\}, \{\lambda \mid \lambda > \alpha_{ij}\}) & \text{if } \alpha_{ij} = 0. \end{cases}$$

Equivalently, the sets $J_i^<$ and J_i^∞ of Proposition 4.12 are empty for all $i \in I$, and so $K^r = \emptyset$ for $r \in [p]$. Observe that this means that $L^r = J$ if $J^r = \emptyset$, which in turn implies $p = r$. Moreover, we also have $\mathcal{H} = \bigoplus_{i \in I} (\mathcal{C}_i \oplus \mathcal{D}_i) = \bigoplus_{i \in I} \mathcal{C}_i$ if \mathcal{H} is a closed conic hemispace, since $J_i^\infty = \emptyset$ implies $\mathcal{D}_i \subseteq \mathcal{C}_i$.

Assume first that $p \geq 2$, which implies $J^1 \neq \emptyset$ as mentioned above. Then, we have $J^2 \cup K^2 \subseteq K^1 = \emptyset$ by (24). It follows that $J^2 = \emptyset$, and so $p = 2$. Thus, we have $I = I^1 \cup I^2$ and $\mathcal{H} = \bigoplus_{i \in I^1 \cup I^2} \mathcal{C}_i$. By Lemma 4.16, the cone $\bigoplus_{i \in I^1} \mathcal{C}_i$ can be represented by

$$(36) \quad \bigoplus_{j \in J^1} \gamma_j x_j \leq \bigoplus_{i \in I^1} \beta_i x_i \text{ and } x_j = 0 \text{ for } j \in L^1 \cup I^2.$$

Note that this is just condition (29), and condition (30) is always satisfied as $J_k^< = J^1$ for all $k \in I^1$. Since $J^2 = \emptyset$, it follows that $\bigoplus_{i \in I^2} \mathcal{C}_i$ is generated by $\{e^i \mid i \in I^2\}$, and then (36) implies that $\mathcal{H} = \bigoplus_{i \in I^1 \cup I^2} \mathcal{C}_i$ is the set of all vectors satisfying

$$(37) \quad \bigoplus_{j \in J^1} \gamma_j x_j \leq \bigoplus_{i \in I^1} \beta_i x_i \text{ and } x_j = 0 \text{ for } j \in L^1,$$

which is a closed homogeneous halfspace. Note that by Lemma 4.16 we arrive at the same conclusion if we assume that $p = 1$ and $J^1 \neq \emptyset$.

Finally, if we assume that $p = 1$ and $J^1 = \emptyset$, then $\mathcal{H} = \bigoplus_{i \in I^1} \mathcal{C}_i$ is generated by $\{e^i \mid i \in I^1 = I\}$, i.e., $\mathcal{H} = \{x \in \mathbb{T}^n \mid x_j = 0 \text{ for } j \in J\}$ is a closed homogeneous halfspace. \square

We now recall an important observation of [2], which will allow us to easily extend the result of Theorem 4.19 to general hemispaces. For the reader's convenience, we give an elementary proof based on tropical segments and their perturbations.

Lemma 4.20 (Briec and Horvath [2]). *Closures of hemispaces = closed hemispaces.*

Proof. In the max-times setting, with usual arithmetics. Consider the closure of a hemisphere \mathcal{H} in $\mathbb{R}_{\max, \times}^n$. Since the closure of a tropical cone is a closed tropical cone (e.g., [4]), we only need to show that the complement of this closure is also a tropical cone. This complement is open, so it consists of all points $x \in \mathbb{C}\mathcal{H}$ for which there exists an open “ball” $B_x^\epsilon := \{u \in \mathbb{R}_{\max, \times}^n \mid |u_i - x_i| < \epsilon \text{ for all } i \in [n]\}$ such that $B_x^\epsilon \subseteq \mathbb{C}\mathcal{H}$. We need to show that if x and y have this property, then any combination $z = \lambda x \oplus \mu y$ with $\lambda \oplus \mu = 1$ also does. If we assume $\lambda = 1$, then

$$z_i = \begin{cases} \mu y_i, & \text{if } \mu y_i > x_i, \\ x_i, & \text{if } \mu y_i \leq x_i. \end{cases}$$

Let us consider $\hat{z} \in \mathbb{R}_{\max, \times}^n$ defined by $\hat{z}_i := z_i + \epsilon_i$, where ϵ'_i are such that $|\epsilon'_i| \leq \epsilon$ for all $i \in [n]$. We can write

$$\hat{z}_i = \begin{cases} \mu y_i + \epsilon_i, & \text{if } \mu y_i + \epsilon_i > x_i \text{ and } x_i < \mu y_i, \\ \mu y_i + \epsilon_i = x_i + \epsilon'_i, & \text{if } \mu y_i + \epsilon_i \leq x_i < \mu y_i, \\ x_i + \epsilon_i, & \text{if } \mu y_i \leq x_i + \epsilon_i \text{ and } \mu y_i \leq x_i, \\ x_i + \epsilon_i = \mu y_i + \epsilon'_i, & \text{if } x_i + \epsilon_i < \mu y_i \leq x_i, \end{cases}$$

where always $|\epsilon'_i| \leq |\epsilon_i| \leq \epsilon$. Thus, defining

$$\begin{cases} \hat{y}_i := y_i + \mu^{-1}\epsilon_i \text{ and } \hat{x}_i := x_i, & \text{if } \mu y_i + \epsilon_i > x_i \text{ and } x_i < \mu y_i, \\ \hat{y}_i := y_i + \mu^{-1}\epsilon_i \text{ and } \hat{x}_i := x_i + \epsilon'_i, & \text{if } \mu y_i + \epsilon_i \leq x_i < \mu y_i, \\ \hat{y}_i := y_i \text{ and } \hat{x}_i := x_i + \epsilon_i, & \text{if } \mu y_i \leq x_i + \epsilon_i \text{ and } \mu y_i \leq x_i, \\ \hat{y}_i := y_i + \mu^{-1}\epsilon'_i \text{ and } \hat{x}_i := x_i + \epsilon_i, & \text{if } x_i + \epsilon_i < \mu y_i \leq x_i, \end{cases}$$

we have $\hat{z} = \mu \hat{y} \oplus \hat{x}$, $\hat{x} \in B_x^\epsilon$ and $\hat{y} \in B_y^{\epsilon''}$, where $\epsilon'' := \mu^{-1}\epsilon$. Since $\mathbb{C}\mathcal{H}$ is tropically convex, it follows that $B_z^\epsilon \subseteq \mathbb{C}\mathcal{H}$ if $B_x^{\epsilon''} \subseteq \mathbb{C}\mathcal{H}$ and $B_y^{\epsilon''} \subseteq \mathbb{C}\mathcal{H}$, proving the claim. \square

Corollary 4.21 (Briec and Horvath [2]). *Closed hemispaces = closed halfspaces.*

Proof. We need to consider the case of a closed halfspace that is not necessarily homogeneous, and of a closed hemisphere that is not necessarily conic. Such a closed halfspace is a set of the form

$$(38) \quad \left\{ x \in \mathbb{T}^n \mid \bigoplus_{j \in J} \gamma_j x_j \oplus \alpha \leq \bigoplus_{i \in I} \beta_i x_i \oplus \delta \text{ and } x_j = 0 \text{ for } j \in L \right\},$$

where I , J and L are pairwise disjoint subsets of $[n]$. As in the conic case, it can be argued that the complement is tropically convex too, so (38) describes a (not necessarily conic) hemisphere.

Conversely, a general closed hemisphere in \mathbb{T}^n can be lifted by means of Theorem 4.7 to a conic hemisphere \mathcal{V} in \mathbb{T}^{n+1} . By Remark 4.8, this hemisphere will be not closed in general, however, if $\overline{\mathcal{V}}$ is its closure, then the section $C_{\overline{\mathcal{V}}}^{\mathbb{1}}$ still coincides with \mathcal{H} . Indeed, for any $z = (\mathbb{1}, x) \in \overline{\mathcal{V}}$ there exists a sequence $\{z^k\}_{k \in \mathbb{N}}$ of vectors of \mathcal{V} such that $\lim_k z^k = z$. Since $z_0 = \mathbb{1}$, taking into account the definition of \mathcal{V} in Theorem 4.7, we can assume that $z^k = (\lambda_k, \lambda_k x^k)$ for some $\lambda_k \in \mathbb{T}$ and $x^k \in \mathcal{H}$. It follows that $\lim_k \lambda_k = \mathbb{1}$ and $\lim_k x^k = x$. Thus, $x \in \mathcal{H}$ because \mathcal{H} is closed. Therefore, we conclude that $C_{\overline{\mathcal{V}}}^{\mathbb{1}} = C_{\mathcal{V}}^{\mathbb{1}} = \mathcal{H}$.

By Theorem 4.19, $\overline{\mathcal{V}}$ can be expressed as a solution set to

$$\bigoplus_{j \in J} \gamma_j x_j \oplus \alpha x_0 \leq \bigoplus_{i \in I} \beta_i x_i \oplus \delta x_0 \text{ and } x_j = 0 \text{ for } j \in L,$$

for some disjoint subsets I , J and L of $[n]$. The original hemisphere in \mathbb{T}^n appears as a section of this closed homogeneous halfspace by $x_0 = \mathbb{1}$, and so it is of the form (38). \square

Corollary 4.22. *Open hemispaces = open halfspaces.*

Proof. Open hemispaces and open halfspaces can be obtained as complements of their closed “partner”. \square

We now characterize general hemispaces by means of specific (P, R) -representations, as foreseen by Theorem 4.7 and Theorem 4.10

Theorem 4.23. *Let $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathbb{T}^n$ be a couple of hemispaces with $0 \in \mathcal{H}_1$. Then, there exist sets I and J satisfying $I + J = [0, n]$ and $0 \in I$, and pairs $(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)})$ of subsets of $\mathbb{T} \cup \{+\infty\}$ of the form (20) for $i \in I$ and $j \in J$ satisfying the rank-one condition (22), such that*

$$(39) \quad \begin{aligned} \mathcal{H}_1 &= \text{conv} \left(\left\{ \lambda e^j \mid j \in J, \lambda \in \alpha_{0j}^{(-)} \right\} \right) \oplus \text{span} \left(\left\{ e^i \oplus \lambda e^j \mid i \in I \setminus \{0\}, j \in J, \lambda \in \alpha_{ij}^{(-)} \right\} \right), \\ \mathcal{H}_2 &= \text{conv} \left(\left\{ \lambda e^j \mid j \in J, \lambda \neq +\infty, \lambda \in \alpha_{0j}^{(+)} \right\} \right) \oplus \text{span} \left(\left\{ e^i \oplus \lambda e^j \mid i \in I \setminus \{0\}, j \in J, \lambda \in \alpha_{ij}^{(+)} \right\} \right). \end{aligned}$$

Moreover, if the pairs $(\alpha_{ij}^{(-)}, \alpha_{ij}^{(+)})$ for $i \in I$ and $j \in J$ satisfy the rank-one condition (22), then the sets \mathcal{H}_1 and \mathcal{H}_2 defined in (39) form a couple of hemispaces.

Proof. The “if” part: with \mathcal{H}_1 and \mathcal{H}_2 given by (39), consider their “homogenizations”:

$$(40) \quad \begin{aligned} \mathcal{V}_1 &= \text{span} \left(\left\{ e^0 \oplus \lambda e^j \mid j \in J, \lambda \in \alpha_{0j}^{(-)} \right\} \right) \oplus \text{span} \left(\left\{ e^i \oplus \lambda e^j \mid i \in I \setminus \{0\}, j \in J, \lambda \in \alpha_{ij}^{(-)} \right\} \right), \\ \mathcal{V}_2 &= \text{span} \left(\left\{ e^0 \oplus \lambda e^j \mid j \in J, \lambda \in \alpha_{0j}^{(+)} \right\} \right) \oplus \text{span} \left(\left\{ e^i \oplus \lambda e^j \mid i \in I \setminus \{0\}, j \in J, \lambda \in \alpha_{ij}^{(+)} \right\} \right). \end{aligned}$$

By Theorem 4.10 (the “if” part), \mathcal{V}_1 and \mathcal{V}_2 form a couple of conic hemispaces. By Proposition 2.6 we have $\mathcal{H}_1 = C_{\mathcal{V}_1}^1$ and $\mathcal{H}_2 = C_{\mathcal{V}_2}^1$, which implies that \mathcal{H}_1 and \mathcal{H}_2 form a couple of hemispaces.

The “only if” part: If \mathcal{H}_1 and \mathcal{H}_2 form a couple of hemispaces, then by Theorem 4.7 they can be represented as sections of some conic hemispaces \mathcal{V}_1 and \mathcal{V}_2 . By Theorem 4.10 (the “only if” part), \mathcal{V}_1 and \mathcal{V}_2 must be as in (40). Using Proposition 2.6, we see that $\mathcal{H}_1 = C_{\mathcal{V}_1}^1$ and $\mathcal{H}_2 = C_{\mathcal{V}_2}^1$ have (P, R) -representations as in (39). \square

5. ACKNOWLEDGEMENT

The authors thank Charles Horvath for useful discussions, and, together with Walter Briec, for sending the full text of their work [2]. Viorel Nitica was partially supported by a grant from Simons Foundation 208729. Sergei Sergeev was supported by EPSRC grant 15735 and partially by RFBR grant 12-01-00886-a and joint RFBR-CNRS 11-01-93106-a.

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